THE EFFECTS OF VERTICAL VARIATION OF BASIC FLOW AND HORIZONTAL GRADIENT OF TEMPERATURE ON LINEAR AND NONLINEAR GRAVITY WAVES

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ABSTRACT

By using two-dimensional dynamical equations in $x-z$ plane with Boussinesq approximation, the effects of the second-order vertical shear of the basic flow ($\overline{U}_{zz}$) and the horizontal gradient of temperature (M) on the gravity wave and the isolated gravity wave are discussed. The magnitudes of $\overline{U}_{zz}$ and M corresponding to the linear and nonlinear stabilities of the gravity waves are worked out, respectively. The results show that amplitude and width of the isolated gravity wave are closely related to $\overline{U}_{zz}$ and M. It is indicated that the isolated gravity wave with a width of about 10 km can be motivated by the disturbance of sub-synoptic scale in the certain ranges of flow field shear and temperature gradient, while the motivated waves may be associated with the cold surge ahead of a cold front and the other mesoscale synoptic systems.

Key words: Boussinesq approximation, isolated gravity wave, cold surge

I. INTRODUCTION

The gravity waves are associated closely with some important synoptic processes, such as the occurrence and development of storm rainfall, typhoon, squall line and the cold surge ahead of a cold front. The results from analysis of global synoptic data show the importance of gravity waves for synoptic processes developing. Thus, meteorologists pay more attention to the problems of instability and development of gravity waves. It is indicated theoretically that the cold surge ahead of a cold front is a gravity wave and its travelling velocity and width can be estimated by using the multiscale method and practical synoptic data. But the effects of the vertical variation of the basic flow (averaged westerly wind) and the horizontal gradient of temperature on the isolated gravity wave have not been concerned yet. In this paper by using the travelling wave method, the problems mentioned above are studied through theoretical analysis and numerical calculation.

II. BASIC EQUATIONS

The following two-dimensional nonlinear equations in $x-z$ plane with Boussinesq approximation are used:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} = -\dfrac{1}{\rho_0}\dfrac{\partial P}{\partial x}, \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z} = -\dfrac{1}{\rho_0}\dfrac{\partial P}{\partial z} + \dfrac{g}{T_0}T, \\ \dfrac{\partial T}{\partial t} + u\dfrac{\partial T}{\partial x} + w\dfrac{\partial T}{\partial z} + (\gamma_d - \gamma)w = 0, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0, \end{cases} \tag{1}$$

where P and T are the disturbed pressure and temperature with respect to those in the case of static state (P_0 and T_0), respectively; and the mark "′" for disturbed variables is omitted. γ is the lapse rate of temperature and γ_d is that for a dry adiabatic process. The other symbols in Eq.(1) are frequently used in meteorology.

In order to discuss the topic conveniently, we suggest the following two schemes.

Scheme A:

$$T = \tilde{T} + T', \quad P = \tilde{P} + P', \quad u = \overline{U}(z) + u', \quad w = w', \tag{2a}$$

where $\tilde{T}$ and $\tilde{P}$ are the zonal mean values in a certain longitude range, and let T=const; u', w', T' and P' are the deviations from zonal mean values of these variables, respectively.

Scheme B:

$$T = \overline{T}(x,z) + T', \quad P = P', \quad u = u', \quad w = w', \tag{2b}$$

where $\overline{T}(x,z)$ stands for the time averaging of T; u', w', T' and P' are the deviations from time mean values of these variables, respectively.

Substituting (2a) into Eq.(1), we obtain

$$\begin{cases} \dfrac{\partial u'}{\partial t} + \overline{U}\dfrac{\partial u'}{\partial x} + u'\dfrac{\partial u'}{\partial x} + w'\dfrac{\partial \overline{U}}{\partial z} + w'\dfrac{\partial u'}{\partial z} = -\dfrac{1}{\rho_0}\dfrac{\partial P'}{\partial x}, \\ \dfrac{\partial w'}{\partial t} + \overline{U}\dfrac{\partial w'}{\partial x} + u'\dfrac{\partial w'}{\partial x} + w'\dfrac{\partial w'}{\partial z} = -\dfrac{1}{\rho_0}\dfrac{\partial P'}{\partial z} + \dfrac{g}{T_0}T', \\ \dfrac{\partial T'}{\partial t} + \overline{U}\dfrac{\partial T'}{\partial x} + w'\dfrac{\partial T'}{\partial z} + (\gamma_d - \gamma)w' = 0. \end{cases} \tag{3a}$$

Again substituting (2b) into Eq.(1),

$$\begin{cases} \dfrac{\partial u'}{\partial t} + u'\dfrac{\partial u'}{\partial x} + w'\dfrac{\partial u'}{\partial z} = -\dfrac{1}{\rho_0}\dfrac{\partial P'}{\partial x}, \\ \dfrac{\partial w'}{\partial t} + u'\dfrac{\partial w'}{\partial x} + w'\dfrac{\partial w'}{\partial z} = -\dfrac{1}{\rho_0}\dfrac{\partial P'}{\partial z} + \dfrac{g}{T_0}T', \\ \dfrac{\partial T'}{\partial t} + u'\dfrac{\partial T'}{\partial x} + w'\dfrac{\partial T'}{\partial z} + u'\dfrac{\partial \overline{T}}{\partial x} + (\gamma_d - \gamma)w' = 0. \end{cases} \tag{3b}$$

Introducing the stream function ψ':

$$u' = -\frac{\partial \psi'}{\partial z}, \qquad w' = \frac{\partial \psi'}{\partial x}, \tag{4}$$

and substituting (4) into (3a),

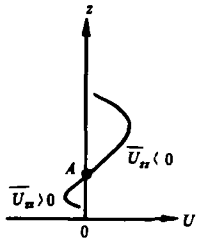

Fig. 1. A pattern of $\overline{U}_{zz}$ in the atmosphere. Above point A, $\overline{U}_{zz}<0$; below A, $\overline{U}_{zz}>0$.

$$\begin{cases} \dfrac{\partial}{\partial t}\Delta\psi' + u'\dfrac{\partial}{\partial x}\Delta\psi' + J(\psi',\Delta\psi') - \overline{U}_{zz}\dfrac{\partial\psi'}{\partial x} - \dfrac{g}{T_0}\dfrac{\partial T'}{\partial x} = 0, \\ \dfrac{\partial T'}{\partial t} + \overline{U}\dfrac{\partial T'}{\partial x} + J(\psi',T') + (\gamma_d - \gamma)\dfrac{\partial\psi'}{\partial x} = 0, \end{cases} \tag{5a}$$

where $\overline{U}_{zz} = \partial^2\overline{U}/\partial z^2$ stands for the second-order vertical shear of zonal mean flow $\overline{U}$. A frequently observed pattern of vertical variation is shown in Fig.1. In Eq.(5a), Δ is two-dimensional Laplace operator and J is Jacobian operator.

Again substituting (4) into (3b), we obtain

$$\begin{cases} \dfrac{\partial}{\partial t}\Delta\psi' + J(\psi',\Delta\psi') = \dfrac{g}{T_0}\dfrac{\partial T'}{\partial x}, \\ \dfrac{\partial T'}{\partial t} + u'\dfrac{\partial T'}{\partial x} + w'\dfrac{\partial T'}{\partial z} + u'\dfrac{\partial \overline{T}}{\partial x} + (\gamma_d - \gamma)w' = 0. \end{cases} \tag{5b}$$

By considering the nonlinear advection terms in Eqs.(5a) and (5b), we obtain

$$\begin{cases} \dfrac{\partial}{\partial t}\Delta\psi' + u'\dfrac{\partial}{\partial x}\Delta\psi' - \dfrac{\partial\psi'}{\partial z}\dfrac{\partial}{\partial x}\Delta\psi' - \overline{U}_{zz}\dfrac{\partial\psi'}{\partial x} - \dfrac{g}{T_0}\dfrac{\partial T'}{\partial x} = 0, \\ \dfrac{\partial T'}{\partial t} + \overline{U}\dfrac{\partial T'}{\partial x} - \dfrac{\partial\psi'}{\partial z}\dfrac{\partial T'}{\partial x} + (\gamma_d - \gamma)\dfrac{\partial\psi'}{\partial x} = 0, \end{cases} \tag{6a}$$

$$\begin{cases} \dfrac{\partial}{\partial t}\Delta\psi' - \dfrac{\partial\psi'}{\partial z}\dfrac{\partial}{\partial x}\Delta\psi' - \dfrac{g}{T_0}\dfrac{\partial T'}{\partial x} = 0, \\ \dfrac{\partial T'}{\partial t} - \dfrac{\partial\psi'}{\partial z}\dfrac{\partial T'}{\partial x} - \dfrac{\partial\psi'}{\partial z}\dfrac{\partial \overline{T}}{\partial x} + (\gamma_d - \gamma)\dfrac{\partial\psi'}{\partial x} = 0. \end{cases} \tag{6b}$$

Both Eqs.(6a) and (6b) can be used to discuss the topic of this paper.

III. THE EFFECTS OF SHEAR FLOW ON LINEAR AND NONLINEAR GRAVITY WAVES

According to travelling wave procedure, letting

$$\begin{bmatrix} \psi' \\ T' \end{bmatrix} = \begin{bmatrix} \Phi \\ T \end{bmatrix}(\theta), \qquad \theta = kx + mz - \omega t, \tag{7}$$

and substituting (7) into (6a), we obtain

$$\begin{cases} (k\overline{U} - \omega)K_n^2\Phi''' - km\Phi' K_n^2\Phi''' - \overline{U}_{zz}k\Phi' - \dfrac{g}{T_0}kT' = 0, \\ (k\overline{U} - \omega)T' - km\Phi' T' + \Gamma k\Phi' = 0, \end{cases} \tag{8}$$

where $\Gamma=\gamma_d-\gamma$; mark "$'$" denotes the derivative of a function with respect to θ, for example, $\Phi'=d\Phi/d\theta$ and $K_n^2=k^2+m^2$.

Eq.(8) can also be written as

$$\begin{cases}(\overline{U}-C_x)K_n^2\Phi'''-mK_n^2\Phi'\Phi'''-\overline{U}_{zz}\Phi'-\dfrac{g}{T_0}T'=0\\ (\overline{U}-C_x)T'-m\Phi'T'+\Gamma\Phi'=0,\end{cases} \tag{9}$$

where $C_x=\omega/k$ is wave velocity along the x-direction. From the second equation of (9), one can see that

$$T'=\frac{\Gamma\Phi'}{(\overline{U}-C_x)-m\Phi'}. \tag{10}$$

Substituting (10) into the first equation of (9), we obtain

$$(\overline{U}-C_x)K_n^2\Phi'''-mK_n^2\Phi'\Phi'''-\overline{U}_{zz}\Phi'+\frac{(g\Gamma/T_0)\Phi'}{(\overline{U}-C_x)-m\Phi'}=0. \tag{11}$$

Furthermore, Eq.(11) can be written as

$$\begin{aligned}\Phi'''&=-\left[\frac{g\Gamma}{T_0[(\overline{U}-C_x)-m\Phi']}-\overline{U}_{zz}\right]\Phi'/K_n^2[(\overline{U}-C_x)-m\Phi']\\ &=-\left[\frac{g}{T_0}\Gamma-\overline{U}_{zz}(\overline{U}-C_x-m\Phi')\right]\Phi'/K_n^2(\overline{U}-C_x-m\Phi')^2.\end{aligned} \tag{12}$$

From $u=-\partial\psi/\partial z$, $w=\partial\psi/\partial x$, then $w=k\Phi'$, $u=-m\Phi'$, the mark "$'$" used here for the disturbed variables is omitted, Eq.(12) therefore becomes

$$w''=-\left[\frac{g}{T_0}\Gamma-\overline{U}_{zz}(\overline{U}-C_x-\frac{m}{k}w)\right]w/K_n^2(\overline{U}-C_x-\frac{m}{k}w)^2. \tag{13}$$

Making the Taylor expansion for Eq.(13) at $w=0$, i.e.

$$w''=-\frac{[g\Gamma/T_0-\overline{U}_{zz}(\overline{U}-C_x-mw/k)]w}{K_n^2(\overline{U}-C_x)^2}\left[1+2\frac{mw/k}{\overline{U}-C_x}+3\frac{(mw/k)^2}{(\overline{U}-C_x)^2}+\cdots\right] \tag{14}$$

and then taking the first-order approximation of (14), we obtain the following linear equation:

$$w''=-\frac{g\Gamma/T_0-\overline{U}_{zz}(\overline{U}-C_x)}{K_n^2(\overline{U}-C_x)^2}w. \tag{15}$$

According to Eq.(15), the effect of shear flow, i.e., the second-order vertical variation of basic zonal wind, on the stability of gravity waves can be discussed as follows.

1. *Linear Stability ofGravity Wave*

The condition of Eq.(15) having the stable solution is $g\Gamma/T_0-\overline{U}_{zz}(\overline{U}-C_x)>0$. Considering that gravity waves would occur frequently in the case of stable stratification of the atmosphere, i.e. $\Gamma>0$, we have

$$\begin{cases}\overline{U}_{zz}<\dfrac{g\Gamma}{T_0}/(\overline{U}-C_x), & \text{when } \overline{U}-C_x>0;\\ \overline{U}_{zz}>\dfrac{g\Gamma}{T_0}/(\overline{U}-C_x), & \text{when } \overline{U}-C_x<0.\end{cases} \tag{16}$$

And the condition of an unstable gravity wave is

$$\begin{cases} \overline{U}_{zz} > \dfrac{g\Gamma}{T_0} / (\overline{U} - C_x), & \text{when } \overline{U} - C_x > 0; \\ \overline{U}_{zz} < \dfrac{g\Gamma}{T_0} / (\overline{U} - C_x), & \text{when } \overline{U} - C_x < 0. \end{cases} \tag{17}$$

By denoting $a = g\Gamma / T_0(\overline{U} - C_x)$, the effect of $\overline{U}_{zz}$ on the linear stability of gravity waves can be deduced: when $-a < \overline{U}_{zz} < a$, the wave is stable. It implies that the smaller values of $|\overline{U}_{zz}|$ is favorable to stability, while in the case of stronger shear flow, such as westerly or easterly jet stream, the instability occurs easily. Moreover, the slowly travelling gravity wave tends to be stable. The magnitude of $\overline{U}_{zz}$ may be estimated as follows: Let $g/T_0 \sim 3\times 10^{-2}$ m^2 s^{-1} ℃$^{-1}$, $\Gamma \sim 4\times 10^{-3}$ ℃ m^{-1}, $\overline{U} - C_x \sim \pm 10$ m s^{-1}, then $a \sim 12\times 10^{-6}$ m^{-1} s^{-1}. As $|\overline{U}_{zz}| < a$, the wave is stable, while $|\overline{U}_{zz}| > a$, the wave is unstable.

It can be seen that the existence of minimum in the vertical distribution of $\overline{U}$ is favorable to the stability of gravity wave. Such a distribution of $\overline{U}$ leads to two vertical circulations near the cold front: one of them is direct circulation located in the lower layer of mid–troposphere, the other is indirect one located in the higher layer.

2. *Nonlinear Stability ofGravity Waves and KdV Equation*

Taking the second–order approximation of Eq.(14), we obtain

$$w'' + \frac{g\Gamma / T_0 - \overline{U}_{zz}(\overline{U} - C_x)}{K_n^2(\overline{U} - C_x)^2} w + \frac{2[g\Gamma / T_0(\overline{U} - C_x) - \overline{U}_{zz}]m / k + \overline{U}_{zz} m / k}{K_n^2(\overline{U} - C_x)^2} w^2 = 0. \tag{18}$$

Nonlinear Eq.(18) is a KdV equation; its special solution is an isolated wave, the stability of which can be tested with the linear stability criterion. For a stable gravity wave, this criterion can be written as

$$w = \frac{3}{2}\frac{R}{q}\operatorname{sech}^2\left(\frac{\sqrt{R}}{2}\theta\right) - \frac{R}{q}, \tag{19}$$

where

$$\theta = kx + mz - \omega t,$$

$$R = \left[\frac{g\Gamma}{T_0} - \overline{U}_{zz}(\overline{U} - C_x)\right] / K_n^2(\overline{U} - C_x)^2,$$

$$q = \left\{2[g\Gamma / T_0(\overline{U} - C_x) - \overline{U}_{zz}]\frac{m}{k} + \frac{m}{k}\overline{U}_{zz}\right\} / K_n^2(\overline{U} - C_x)^2.$$

The abruptness of this isolated wave is described by $\mu = \sqrt{R}/2$.

The solution of an unstable isolated gravity wave is

$$w = \frac{3}{2}\frac{R}{q}\operatorname{sech}^2\left(\frac{\sqrt{-R}}{2}\theta\right). \tag{20}$$

From Eqs.(19) and (20), one can discuss the stability of gravity waves as follows:

(1) *Effect of $\overline{U}_{zz}$ on the strength ofthe gravity wave*

As known from (19), the amplitude of the isolated wave "A" is determined by

$$A=\frac{3}{2}\frac{R}{q}=\frac{3}{2}\frac{k}{m}\frac{g\Gamma/T_0-\overline{U}_{zz}(\overline{U}-C_x)}{2g\Gamma/T_0(\overline{U}-C_x)-\overline{U}_{zz}}. \tag{21}$$

Given the horizontal and vertical wave numbers and the stratification parameter Γ, expression (21) becomes

$$A\propto(\overline{U}-C_x)\frac{1}{1+\dfrac{g\Gamma/T_0}{g\Gamma/T_0-\overline{U}_{zz}(\overline{U}-C_x)}}. \tag{22}$$

Consider the atmospheric stratification stable in order to analyse conviniently the effect of $\overline{U}_{zz}$ on the gravity wave, as illustrated in Fig.2. It is shown that in the easterly jet region where $\overline{U}_{zz}>0$ as in Fig.2b, when $\overline{U}-C_x<0$, the larger the $\overline{U}_{zz}$ is, the smaller the amplitude is; while when $\overline{U}-C_x>0$, the smaller the $\overline{U}_{zz}$ is, the larger the amplitude is. In the easterly jet region where $\overline{U}_{zz}<0$ as in Fig.2a, the results can be worked out in the same way.

It is indicated that for the gravity wave with $C_x<\overline{U}$, the stronger easterly jet makes the isolated gravity wave decay, while the weaker one makes it intensify. Melvile (1988) verified the relation between the intensities of squall lines and the easterly jet by a numerical experiment, and indicated that the strong easterly jet makes squall lines decay and the weak one makes it intensify. His conclusion seems to be in accordance with our results about the relation between amplitude of the isolated gravity wave and $\overline{U}_{zz}$. Given the characteristic values of some variables in formula (21), this relation can be estimated quantitatively as follows. Let the ratio of vertical to horizontal wave lengths $k/m\sim10^{-2}$, $g/T_0\sim3\times10^{-2}$ m s^{-2} ℃$^{-1}$, $\Gamma\sim4\times10^{-3}$ ℃ m^{-1}, $\overline{U}-C_x\sim\pm30$ m s^{-1}, then the estimated results are shown in Table 1.

It can be seen from Table 1 that in the jet region where $\overline{U}-C_x>0$, the strong westerly jet makes the isolated gravity wave intensify; while the strong easterly jet makes it decay. In the case of $\overline{U}-C_x<0$, there is the contrary situation. Therefore, the conditions of development of the isolated gravity wave associate not only with the vertical variation of the basic flow, but also with the basic flow itself and the travelling velocity of the gravity wave.

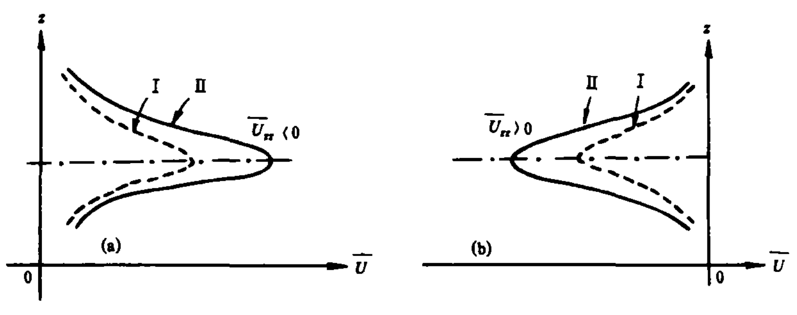

Fig. 2. The relation between jet distribution and amplitude of the gravity wave under different wave velocity. (a) For the westerly jet: when $C_x<\overline{U}$, curve I is favorable to the decrease of the wave amplitude and curve II is favorable to the increase; when $C_x>\overline{U}$, there is the contrary results; (b) For the easterly jet: same as (a).

Table 1. The Relation between the Amplitude of the Isolated Gravity Wave A (m s^{-1}) and the Vertical Variation of the Basic Flow $\overline{U}_{zz}$ (m^{-1} s^{-1}) under Given Values of Difference $\overline{U}-C_x$ (m s^{-1})

$\overline{U}-C_x$	$\overline{U}_{zz}$				
	3×10^{-6}	1×10^{-6}	0	-1×10^{-6}	-3×10^{-6}
30	0.09	0.10	0.25	0.25	0.29
−30	0.29	0.25	0.25	0.19	0.09

(2) *Effect of $\overline{U}_{zz}$ on the width of isolated gravity wave*

It can be seen from Eq.(19) that the wave abruptness parameter is $\mu=\sqrt{R}/2$, and the wave width is $1/\mu$, where $R=[g\Gamma/T_0-\overline{U}_{zz}(\overline{U}-C_x)]/K_n^2(\overline{U}-C_x)^2$. For the stable wave, therefore, when $\overline{U}-C_x>0$, the larger the $\overline{U}_{zz}$ is, the longer the wave width is, and vice versa; when $\overline{U}-C_x<0$, there is the contrary situation. The results, according to the given vertical distributions of the basic flow shown in Fig.2, can be summarized as follows. In the case of the westerly jet (see Fig.2a), when $C_x<\overline{U}$, curve I corresponds to the wide wave, while curve II corresponds to the narrow wave, and on the contrary when $C_x>\overline{U}$. In the case of the easterly jet (see Fig.2b), there is the corresponding situation.

Now, the relation between wave width and $\overline{U}_{zz}$ can be estimated as follows. Let $\overline{U}_{zz}^{\bullet}=g\Gamma/T_0(\overline{U}-C_x)$, $\overline{U}_{zz}=\overline{U}_{zz}^{\bullet}+\delta_1$, $R=-\delta_1/K_n^2(\overline{U}-C_x)$, $K_n^2\sim3.6\times10^{-7}$, and values of the other variables in formula (19) are the same as in Table 1, the estimated results of wave width $(1/\mu)$ related to δ_1 are shown in Table 2.

Table 2. Relation between Wave Width $1/\mu$ (km) and Vertical Variation of the Basic Flow δ_1 (m^{-1} s^{-1})

δ_1	0	-3.6×10^{-14}	-3.6×10^{-12}	-3.6×10^{-10}	-3.6×10^{-8}
$1/\mu$	∞	200	20	2	0.2

In Table 2, $\overline{U}-C_x<0$. If $\overline{U}-C_x>0$, δ_1 is positive, and the corresponding result can be obtained. As known from Table 2, when the values of vertical variation of the basic flow $\overline{U}_{zz}\sim g\Gamma/T_0(\overline{U}-C_x)\pm3.6\times10^{-12}$ m^{-1} s^{-1}, the width of the isolated gravity wave is about 20 km, which corresponds to the cold surge ahead of a cold front.

IV. EFFECT OF HORIZONTAL GRADIENT OF TEMPERATURE ON THE LINEAR AND NONLINEAR GRAVITY WAVES

Substituting (7) into (6b), we obtain

$$\begin{cases} C_xK_n^2\Phi'''+m\Phi'\Phi'''K_n^2+gT'/T_0=0, \\ (C_x+m\Phi')T'+M\Phi'm/k-\Gamma\Phi'=0, \end{cases} \tag{23}$$

where the gradient of averaged temperature in the x-direction is denoted by $M=\partial\overline{T}/\partial x$. From the second equation of (23), the expression for T' can be found and then substitute it into the first one,

$$\Phi''' = -\frac{(\Gamma\Phi' - M\Phi' m / k)g / T_0}{(C_x + m\Phi')^2 K_n^2}. \tag{24}$$

Since $w = k\Phi'$, (24) can also be written as

$$w'' = -\frac{(\Gamma - Mm / k)wg / T_0}{C_x^2 K_n^2 (1 + mw / C_x k)^2}. \tag{25}$$

Making Taylor expansion for (25) at $w = 0$,

$$w'' = -\frac{g}{T_0 C_x^2 K_n^2}(\Gamma - mM / k)w + \frac{g}{T_0 C_x^2 K_n^2}(\Gamma - mM / k)\frac{2m}{C_x k} w^2 + \cdots \tag{26}$$

Expression (26) is the basic equation used to discuss the effect of the horizontal gradient of temperature on the gravity wave.

1. *Linear Stability of Gravity Wave*

Taking the first-order approximation of Eq.(26), we obtain

$$w'' + \frac{g}{T_0 C_x^2 K_n^2}(\Gamma - \frac{m}{k} M)w = 0. \tag{27}$$

It can be seen from (27) that if the gravity wave is stable, $\Gamma - mM / k > 0$ is required, and vice versa. The stability of the gravity wave depends on Γ when $M = 0$; this is a well-known fact in meteorology. When $M \neq 0$ it can be estimated as follows: $\Gamma > mM / k$ if the wave is stable; moreover, assuming $m / k > 0$, then $\Gamma k / m > M$. Taking $\Gamma \sim 4 \times 10^{-3}$ ℃ m^{-1} and $k / m \sim 10^{-2}$, then the gravity wave is stable when $M < 4$℃ / 100 km. Generally, if we take the effect of temperature gradient on the stability of gravity waves into account, then the sub-synoptic and mesoscale gravity waves satisfy easily the conditions of stability. The synoptic scale gravity wave is stable when $M < 0.4$℃ / 100 km and unstable when $M > 0.4$.

2. *Nonlinear Instability of Gravity Wave and KdV Equation*

Taking the second-order approximation of Eq.(26), we have

$$w'' + \frac{g}{T_0 C_x^2 K_n^2}(\Gamma - mM / k)w - \frac{g}{T_0 C_x^2 K_n^2}(\Gamma - mM / k)\frac{2m}{C_x k} w^2 = 0. \tag{28}$$

The stability near the equilibrium point of Eq.(28) can be investigated with the view point of linear stability. Eq.(28) is a KdV equation, of which the special solution is an isolated wave. If $\Gamma - mM / k > 0$, the solution in form of the stable isolated gravity wave is

$$w = -\frac{3}{4}\frac{k}{m} C_x \mathrm{sech}^2\left(\frac{\sqrt{R}}{2}\theta\right) + \frac{C_x}{2}\frac{k}{m}, \tag{29}$$

where θ is defined as the same as in Eq.(19), and $R = g(\Gamma - mM / k) / T_0 C_x^2 K_n^2$; if $\Gamma - mM / k < 0$, the solution in form of the unstable isolated gravity wave is

$$w = \frac{3}{4}\frac{k}{m} C_x \mathrm{sech}^2\left(\frac{\sqrt{-R}}{2}\theta\right). \tag{30}$$

As known from Eqs.(29) and (30), the wave amplitude is directly proportional to C_x, so the faster the wave travels, the greater its amplitude is. The stable isolated gravity wave moving eastward is the wave ridge, and that moving westward is the wave trough; while the unstable wave moving eastward belongs to the wave trough pattern, and that moving westward belongs to the wave ridge pattern.

Now, discuss the relation of wave width to the horizontal gradient of temperature. As mentioned above, wave width $1/\mu=2/\sqrt{R}$, where R is directly proportional to $g(\Gamma-mM/k)$, and inversely proportional to $T_0 C_x^2 K_n^2$. Given temperature stratification ($\Gamma>0$), the ratio of wave numbers ($m/k>0$), T_0 and C_x, one can see from calculation that when the isolated gravity wave moves from the warmer region to the colder region, its width decreases; moreover, the larger the temperature difference is, the smaller its width is; and when the wave moves from the colder region to the warmer region, there is the contrary situation. The magnitude of wave width can be estimated as follows. Taking $C_x \sim 20$ m s^{-1}, $g/T_0 \sim 3\times10^{-2}$ m ℃$^{-1}$ s^{-2}, $\Gamma\sim 4\times 10^{-3}$ ℃ m^{-1}, $k/m \sim 10^{-2}$, wave length in the x-direction $L_x \sim 10^5$ m and that in the z-direction $L_z \sim 10^3$ m, $M=(\partial\overline{T}/\partial x)^* - \delta_2$, where $(\partial\overline{T}/\partial x)^* = \Gamma k/m$, the numerical relation of wave width $1/\mu$ to temperature gradient δ_2 is shown in Table 3.

Table 3. Relation of the Isolated Gravity Wave Width $1/\mu$ (km) to the Horizontal Temperature Gradient of δ_2 (℃ km^{-1})

δ_2	0	1.2×10^{-4}	1.2×10^{-2}	1.2×10^{0}
$1/\mu$	∞	10	1	0.1

As shown in Table 3, if the horizontal gradient of temperature is large enough ($M\sim 4\times(10^{-2} \sim 10^{-3})$ ℃ km^{-1}), the isolated gravity wave with a width of about 10 km can be generated. According to weather analysis, such a gradient can exist near a cold front, which is favorable to the development of the wave in form of a cold surge.

V. CONCLUSION

Many factors play important roles in gravity wave developing, so that it is a complex problem. For simplicity, the two-dimensional equations with Boussinesq approximation are used in this paper. Applying the travelling wave method, the effects of vertical variation of the basic flow $\overline{U}_{zz}$ and horizontal gradient of temperature M on the linear and nonlinear gravity waves are discussed. The magnitudes of $\overline{U}_{zz}$ and M corresponding to linear stability are estimated. The effects of $\overline{U}_{zz}$ and M on the nonlinear isolated gravity wave are analysed in detail and estimated quantitatively by taking characteristic values of some variables in the equations. The results show that the development of the isolated gravity wave with a width of about 10 km (such as the cold surge ahead of a cold front) corresponds to magnitudes of $\overline{U}_{zz} \sim g\Gamma/T_0(\overline{U}-C_x)$ 10^{-11} m^{-1} s^{-1} and $M\sim \Gamma k/m$ $(10^{-2}\sim 10^{-3})$ ℃ km^{-1}. In principle, this method can also be used to study the conditions of gravity wave developing in the other mesoscale synoptic systems and estimate the corresponding values of $\overline{U}_{zz}$ and M. Indeed, it is very useful for the weather analysis and forecasting.

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