LOCAL POTENTIAL AND PRINCIPLE OF VORTICITY CONSERVATION

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ABSTRACT

In this paper, the local potential of nondivergent flow on beta plane is derived from the motion equation. The conservation principle for vorticity is obtained when the local potential tends to be minimum with the variational technique. Some other properties such as Lagrangian of vorticity equation, energy etc. are also discussed briefly in the paper.

Key words: local potential, vorticity conservation, motion equation

I. INTRODUCTION

Whitham et al. attempted to find the equivalent variational problem of the conservation principle for vorticity and then to study the dynamical properties of the waves by means of the average Lagrange approach. Seliger and Whitham (1968) derived the Lagrangian of the linear vorticity equation, but failed for nonlinear one. Wu (1987) obtained Lagrangian by means of the restrictive variational principle proposed by Finlayson (1972). However, the method is rather flexible and physically, its principle is not quite clear. In this paper, we try to start from the motion equation instead of the vorticity equation to find the variational problem with the local potential method proposed by Glansdoff and Prigogine (1971). Physically, the method is rather elegant. In the paper, the variational problem for barotropic, nondivergent vorticity equation is studied. In addition, the effect of orography is also discussed. Finally, the physical meaning of the Lagrangian obtained is explained.

II. LOCAL POTENTIAL FOR BAROTROPIC NONDIVERGENT FLOW

The motion equation of barotropic, nondivergent flow may be written as

$$\frac{\partial u}{\partial t}-\zeta_a v=-\frac{\partial \Phi}{\partial x}, \tag{1a}$$

$$\frac{\partial v}{\partial t}+\zeta_a u=-\frac{\partial \Phi}{\partial y}, \tag{1b}$$

where u, v are the velocity components in x,y directions, respectively, and

$$\zeta_a=\zeta+f, \quad \zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}, \tag{2}$$

Φ is the total energy and can be expressed as

$$\Phi=\phi+\frac{1}{2}(u^2+v^2), \tag{3}$$

where ϕ is geopotential. On β-plane, f reads

$$f=f_0+\beta y, \tag{4}$$

where $\beta=df/dy$, and it is taken as constant.

If the motion is nondivergent horizontally, then

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{5}$$

Thus a stream function ψ can be introduced, such that

$$u=-\frac{\partial \psi}{\partial y}, \quad v=\frac{\partial \psi}{\partial x}. \tag{6}$$

Substituting Eqs. (4) and (6) into Eq. (1), the motion equation is rewritten as

$$\frac{\partial u}{\partial t}=-\frac{\partial}{\partial x}(\Phi-\zeta_a\psi)-\psi\frac{\partial \zeta_a}{\partial x}, \tag{7a}$$

$$\frac{\partial v}{\partial t}=-\frac{\partial}{\partial y}(\Phi-\zeta_a\psi)-\psi\frac{\partial \zeta_a}{\partial y}, \tag{7b}$$

or

$$\frac{\partial \mathbf{V}}{\partial t}=-\nabla(\Phi-\zeta_a\psi)-\psi\nabla\zeta_a. \tag{7c}$$

This is a nonlinear equation which has taken into account the β-effect as well as nondivergent condition. The advantages of the equation were discussed by Wu (1987).

If the reference state is ψ_0, then the velocity $\mathbf{V}$ can be expressed as

$$\mathbf{V}=\mathbf{V}_0+\delta\mathbf{V}. \tag{8}$$

The subscript "0" denotes the reference state and δ fluctuation thereafter. In general, any physical quantity such as $\psi\nabla\zeta_a$ can be expressed as

$$\psi\nabla\zeta_a=(\psi\nabla\zeta_a)_0+\delta(\psi\nabla\zeta_a). \tag{9}$$

With this expression, Eq. (7) reads

$$\frac{\partial\delta\mathbf{V}}{\partial t}=-\frac{\partial \mathbf{V}_0}{\partial t}-\nabla(\Phi-\psi\zeta_a)_0-(\psi\nabla\zeta_a)_0-\delta\nabla(\Phi-\psi\zeta_a)-\delta(\psi\nabla\zeta_a). \tag{10}$$

The assumption that ∇ and δ are exchangeable has been used to get Eq. (10), which implies that the fluctuation of the flow is continuously differentiable.

Multiplying $\delta\mathbf{V}$ by Eq. (10), we can obtain

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2}|\delta\mathbf{V}|^2=&-\frac{\partial \mathbf{V}_0}{\partial t}\cdot\delta\mathbf{V}-\delta\mathbf{V}\cdot\nabla(\Phi-\psi\zeta_a)_0\\&-\delta\mathbf{V}\cdot(\psi\nabla\zeta_a)_0-\delta\mathbf{V}\cdot\delta(\psi\nabla\zeta_a)\\&-\delta\mathbf{V}\cdot\delta\nabla(\Phi-\psi\zeta_a).\end{aligned} \tag{11}$$

Since

$$\delta \mathbf{V} \cdot \nabla (\Phi - \psi \zeta_a) = \nabla \cdot [\delta \mathbf{V} (\Phi - \psi \zeta_a)] - (\Phi - \psi \zeta_a) \nabla \cdot \delta \mathbf{V}, \tag{12}$$

with nondivergent condition $\nabla \cdot \delta \mathbf{V} = 0$, we have $\delta \mathbf{V} \cdot \nabla (\Phi - \psi \zeta_a) = \nabla \cdot [\delta \mathbf{V} (\Phi - \psi \zeta_a)]$. Integrating Eq. (11) over whole region, then we have

$$\frac{\partial}{\partial t} \int \frac{1}{2} |\delta \mathbf{V}|^2 d\tau = -\int \frac{\partial \mathbf{V}_0}{\partial t} \cdot \delta \mathbf{V} d\tau - \int \delta \mathbf{V} \cdot (\psi \nabla \zeta_a)_0 d\tau - \int \delta \mathbf{V} \cdot \delta (\psi \nabla \zeta_a) d\tau. \tag{13}$$

If the fluctuation is small in the neighbourhood of the reference state, then $\delta \mathbf{V} \cdot \delta (\psi \nabla \zeta_a)$ is much smaller and is negligible. Thus Eq. (13) is reduced approximately to

$$\frac{\partial}{\partial t} \int \frac{1}{2} |\delta \mathbf{V}|^2 d\tau = -\int \frac{\partial \mathbf{V}_0}{\partial t} \cdot \delta \mathbf{V} d\tau - \int \delta \mathbf{V} \cdot (\psi \nabla \zeta_a)_0 d\tau. \tag{14}$$

Integrating Eq. (14) with respect to time, we have

$$\int \frac{1}{2} |\delta \mathbf{V}|^2 d\tau = -\delta \int \left[\frac{\partial \mathbf{V}_0}{\partial t} + (\psi \nabla \zeta_a)_0 \right] \cdot \mathbf{V} d\tau dt \geqslant 0. \tag{15}$$

Let

$$F(\psi, \psi_0) = -\int \left[\frac{\partial \mathbf{V}_0}{\partial t} + (\psi \nabla \zeta_a)_0 \right] \cdot \mathbf{V} d\tau dt. \tag{16}$$

Then the right hand side of Eq. (15) is equal to δF. The quantity $F(\psi, \psi_0)$ can be considered as a functional of two variables.

The increment of F around the reference state is

$$\begin{aligned} \Delta F &= F(\psi, \psi_0) - F(\psi_0, \psi_0) \\ &= \int \left[\frac{\partial \mathbf{V}_0}{\partial t} + (\psi \nabla \zeta_a)_0 \right] \cdot (\mathbf{V}_0 - \mathbf{V}) d\tau dt. \end{aligned} \tag{17}$$

At the reference state, $\psi = \psi_0$, i.e. $\mathbf{V} = \mathbf{V}_0$, thus

$$\Delta F = 0. \tag{18}$$

From Eq. (15), we know that when the state deviates the reference state, i.e. $\psi \neq \psi_0$, then

$$\Delta F > 0. \tag{19}$$

As far as the whole system is concerned, in the neighbourhood of the reference state, the following relation holds:

$$\int [F(\psi, \psi_0) - F(\psi_0, \psi_0)] d\tau dt > 0. \tag{20}$$

This property of $F(\psi, \psi_0)$ is schematically shown in Fig. 1. Fig. 1 indicates that

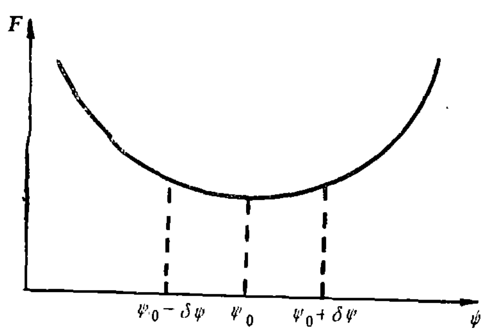

Fig. 1. Schematic diagram of $F(\psi, \psi_0)$.

$F(\psi, \psi_0)$ has the feature: for $\delta\psi$

$$\left.\frac{\delta F}{\delta \psi}\right|_{\psi=\psi_0} \cdot \tag{21}$$

This shows that $F(\psi, \psi_0)$ is minimum when $\psi=\psi_0$. If $\mathbf{V}$, the velocity is expressed by the steam function, then the Euler-Lagrange equation is obtained

$$\left.\frac{\delta F}{\delta \psi}\right|_{\psi=\psi_0} = \frac{\partial}{\partial x}\left.\left(\frac{\partial^2 \psi_0}{\partial t \partial y} + \psi_0 \frac{\partial \zeta_{a0}}{\partial y}\right)\right|_{\psi=\psi_0} + \frac{\partial}{\partial y}\left.\left(\frac{\partial^2 \psi_0}{\partial t \partial x} - \psi_0 \frac{\partial \zeta_{a0}}{\partial x}\right)\right|_{\psi=\psi_0} = 0. \tag{22}$$

After manipulating, Eq. (22) may be written as

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi + f) = 0. \tag{23}$$

This is the conservation principle of vorticity for nondivergent fluid.

Glansdorff and Prigogine (1971) called the functional with the properties of Eqs. (19) and (22) as local potential. With the motion equation and local potential, the vorticity equation can be derived by variational method. In other words, this implies that the conservation principle of vorticity is related to the minimization of the local potential.

Similarly, the aforementioned method can be easily used to the steady case. The local potential is

$$F(\psi, \psi_0) = -\int (\psi \nabla \zeta_a)_0 \cdot \mathbf{V} d\tau dt. \tag{24}$$

When $\psi=\psi_0$, $F(\psi, \psi_0)$ reaches minimum, that is

$$\left.\frac{\delta F}{\delta \psi}\right|_{\psi=\psi_0} = 0.$$

Thus the Euler-Lagrange equation can be expressed as

$$J(\psi, \nabla^2\psi + f) = 0. \tag{25}$$

This is the conservation principle of vorticity in the steady case.

III. LOCAL POTENTIAL INCLUDING TOPOGRAPHIC EFFECT

The above-mentioned method can also be generalized to the case that the topographic effect is included. For the homogeneous, incompressible fluid, the continuity equation is

$$\nabla \cdot (D-h)\mathbf{V} = 0, \tag{26}$$

where D is the depth of water, and h the height of the topography. $\mathbf{V}$ can be written as

$$\mathbf{V} = \left(1-\frac{h}{D}\right)^{-1} \mathbf{k} \times \nabla\psi. \tag{27}$$

Substituting Eq. (27) into the motion equation, obtain

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[\Phi - \zeta_a\left(1-\frac{h}{D}\right)^{-1}\psi\right] - \psi\frac{\partial}{\partial x}\left[\zeta_a\left(1-\frac{h}{D}\right)^{-1}\right], \tag{28a}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}\left[\Phi - \zeta_a\left(1-\frac{h}{D}\right)^{-1}\psi\right] - \psi\frac{\partial}{\partial y}\left[\zeta_a\left(1-\frac{h}{D}\right)^{-1}\right]. \tag{28b}$$

Similar to the approach described in Section II, the local potential is expressed as

$$\tilde{F}(\psi, \psi_0) = -\int\left[\frac{\partial \mathbf{V}_0}{\partial t} + \psi_0 \nabla\left(\zeta_a\left(1-\frac{h}{D}\right)^{-1}\right)_0\right] \cdot \mathbf{V} d\tau dt, \tag{29}$$

From the minimum condition of the local potential, i.e.,

$$\left.\frac{\delta \tilde{F}}{\delta \psi}\right|_{\psi=\psi_0}=0, \tag{30}$$

the expression

$$\frac{\partial \zeta}{\partial t}+J\left(\psi, (\nabla^2\psi+f)\left(1-\frac{h}{D}\right)^{-1}\right)=0 \tag{31}$$

is derived, where $\zeta=(\partial v/\partial x)-(\partial u/\partial y)$. By means of Eq. (27), ζ can be written as

$$\zeta=\frac{\partial}{\partial x}\left[\left(1-\frac{h}{D}\right)^{-1}\frac{\partial \psi}{\partial x}\right]+\frac{\partial}{\partial y}\left[\left(1-\frac{h}{D}\right)^{-1}\frac{\partial \psi}{\partial y}\right]. \tag{32}$$

Hence, Eq. (31) is the voticity equation when the topographic effect is taken into consideration.

IV. LAGRANGIAN AND LOCAL POTENTIAL

Whitham derived the Lagrangian for the linear vorticity equation by means of the following procedures.

The linear vorticity equation may be written as

$$\nabla^2\frac{\partial \psi}{\partial t}+\beta\frac{\partial \psi}{\partial x}=0. \tag{33}$$

Let

$$\psi=\frac{\partial F}{\partial t}. \tag{34}$$

Then Eq. (33) is reduced to

$$\nabla^2\frac{\partial^2 F}{\partial t^2}+\beta\frac{\partial^2 F}{\partial t\partial x}=0. \tag{35}$$

It is easy to verify that this equation is equivalent to minimizing the functional J in the form

$$J=\iint\left[\left(\frac{\partial^2 F}{\partial x\partial t}\right)^2+\left(\frac{\partial^2 F}{\partial y\partial t}\right)^2-\beta\frac{\partial F}{\partial x}\frac{\partial F}{\partial t}\right]d\tau dt. \tag{36}$$

In other words, the Euler-Lagrange equation of Eq. (36) is Eq. (35). From Eq. (36), we know that the Lagrangian is

$$L=\frac{1}{2}\left[\left(\frac{\partial^2 F}{\partial x\partial t}\right)^2+\left(\frac{\partial^2 F}{\partial y\partial t}\right)^2-\beta\frac{\partial F}{\partial x}\frac{\partial F}{\partial t}\right]. \tag{37}$$

Physically, both the first and the second term on the right side of Eq. (37) are kinetic energy, which may be proved directly by Eq. (34). The third term is the potential energy of wave, which was verified by Bachwald (1972).

For the mean flow, the motion equation may be written as

$$\frac{\partial \mathbf{V}_0}{\partial t}=-\nabla(\Phi-\psi\zeta_a)_0-(\psi\nabla\zeta_a)_0, \tag{38}$$

or

$$\frac{\partial \mathbf{V}_0}{\partial t}+(\psi\nabla\zeta_a)_0=-\nabla(\Phi-\psi\xi_a)_0. \tag{39}$$

Substituting this relation into Eq. (16), the local potential is obtained

$$F(\psi,\psi_0)=\int\nabla(\Phi-\psi\zeta_a)_0\cdot\mathbf{V}d\tau dt. \tag{40}$$

Since $\nabla(\Phi-\psi\zeta_a)_0$ is a conservative force, Eq. (40) then expresses the total work done by **the conservative force.** $(\Phi-\psi\zeta_a)_0$ can be written as

$$(\Phi-\psi\zeta_a)_0=\phi_0+\frac{1}{2}(u_0^2+v_0^2)-(\psi\zeta_a)_0, \tag{41}$$

where ϕ_0 is the potential energy, $(1/2)(u_0^2+v_0^2)$ the kinetic energy, and $(\psi\zeta_a)_0$ is one kind of energy caused by vortex motion and called as vortex energy. Thus the local potential is the total energy caused by the gradient forces of energy. In some sense, it is similar to Lagrangian.

From the above discussion, it can be seen that the processes to find the extremum by means of Eq. (16) correspond to those to operate curl to the motion equation. In other words, the processes to derive the vorticity equation, in a physical sense, is those of minimization of the local potential.

In terms of the local potential, we can find the approximate solution with minimal errors. With Prigogine's method and principle of functional extremum, we choose the basic function such that the coefficients of the approximate solution may be determined by the way of minimazing errors. To some extent, this approach is similar to the truncated spectral method widely used in meteorology. However, the spectral method truncates to several terms by assumtion or by experiences while the local potential approach contains the minimization of the error.

An illustration of the method is given by the following simple example. Let

$$\psi=\sum_{n=-\infty}^{\infty}A_n(t)\psi_n,\quad \psi_0=\sum_{n=-\infty}^{\infty}A_n^0(t)\psi_n, \tag{42}$$

where ψ_n is the element of coordinate which in general may be represented as a complete system of orthogonal function or some special functions. In this connection, we may refer to the Galerkin's method.

In practive, n in taken as a finite number. The problem is how to select such coefficients $A_n(t)$ that the total error of the solution tends to be minimum. Assuming

$$\psi_n=e^{i\theta_n},\quad \theta_n=k_nx+m_ny, \tag{43}$$

where θ_n is the phase of the nth wave, k_n and m_n are wavenumbers in x, y directions, respectively. When ψ_n is a real function,

$$A_n=-A_{-n},\qquad A_n=A_{-n}^*. \tag{44}$$

In terms of orthogonal condition, i.e.,

$$\int\psi_n\psi_m d\tau=\begin{cases}0, & n\neq m\\ 1, & n=m\end{cases} \tag{45}$$

and the conjugate condition

$$\int\psi_n\psi_m\psi_e d\tau=\begin{cases}0, & \theta_n+\theta_s+\theta_e\neq 0\\ 1, & \theta_n+\theta_s+\theta_e=0\end{cases} \tag{46}$$

the local potential can be expressed as

$$F(\psi,\psi_0)=\sum_S[\dot{A}_s^0(m_s^2+k_s^2)-\beta ik_sA_s^0$$

$$+\sum_{p}\sum_{q}(m_s k_p - m_p k_s)(m_s^2 + k_s^2)A_q^0 A_p^0]A_s. \tag{47}$$

The process of finding extremum of $F(\psi, \psi_0)$ is equivalent to finding A_n by Eq. (47), i.e.,

$$\frac{\partial F}{\partial A_s} = 0, \quad A_s = A_s^0. \tag{48}$$

This has been discussed in detail by Glansdoff and Prigogine (1971).

By means of Eqs. (47) and (48), we obtain

$$(k_s^2 + m_s^2)\dot{A}_s = ik_s\beta A_s - \sum_{p}\sum_{q}(m_s k_p - m_p k_s)(m_s^2 + k_s^2)A_p A_q, \tag{49}$$

where p and q are the indexes making Eq. (46) hold.

The above process of finding $\dot{A}_s$ is similar to that of taking the truncated finite terms in Eq. (42) and then substituting them into Eq. (23). However, it includes the process of finding the variational extremum while the truncated spectral method is just of experience.

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