A SPECTRUM OF LYAPUNOV EXPONENTS OBTAINED FROM A CHAOTIC TIME SERIES

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ABSTRACT

A complete spectrum of Lyapunov exponents (LEs) is obtained from 1970— 1985 daily mean pressure measurements at Shanghai by means of a correlation matrix analysis technique and it is found that there exist LEs>0, and <0, with their sum < zero ($\sum \lambda_i < 0$), thus showing the evolution of the climate–weather system represented by the series to be chaotic.

The sum of positive LE is known to represent the bodily divergence of the system and the sum of these positive LEs is theoretically found to be Kolmogorov entropy of the system. This paper shows that with the time lag $\tau = 5$, the parameter $m = 2$ and the dimensionality $d_M = 9$, the sum of the positive LEs $\sum_{\lambda_i > 0} \lambda_i = K = 0.110405$ whereupon $T = 1 / K = 9$ is obtained as the predictable time scale, a result close to that acquired by the dynamic–statistical approach in early days and also in agreement with that present by the authors themselves(1991).

Key words: spectrum of Lyapunov exponents, chaotic time series, daily mean pressure

I. INTRODUCTION

An attractor is found to be in phase space in the evolution of a short–term climate system based on 1870—1980 monthly mean temperature series at Shanghai and Guangzhou in terms of phase space continuation and the attractor is of fractional dimensionality, indicating that the evolution is chaotic in nature(Yan et al., 1991). Also, they showed that the attractor in the related phase is of fractional dimensionality and the acquired Kolmogorov entropy is positive in the 1970— 1985 series of daily mean pressure for Shanghai employed in the investigation of short–term weather change(Yan et al., 1991), thereby indicating that the evolution of the short–range weather system embodied by the series is a motion of chaotic nature.

An attractor of fractional dimensionality formed in the phase space of a chaotic motion is referred to as a strange attractor. One important aspect in the study of a chaotic motion is to explore the geometric properties of expansion and contraction in a given direction of the phase space of the system, and the physical quantity describing such properties is the Lyapunov exponents (LEs), which are the long–term mean of the expansion or contraction around the orbit in a given direction. For the direction with LE<0, the phase volume is contracted, leading to a steady orbit of the motion, insensitive to initial conditions. For a dissipative system the phase space is contractive, on the whole, so that the sum of the LEs should be <0. On the other hand, the system experiences continuous expansion in the direction with LE>0 such that two state points getting closer will be increasingly apart, or unrelated, which will lead to the unpredictability of the long–term evolution ahead. For a strange attractor at least one of the LEs should be positive and for LE$=0$ the related initial error is neither amplified nor lessened.

From recent contributions on the extraction of LE out of one–dimensional(1D) time series one can see that there are dominantly two schemes in use. One scheme was presented by Wolf(1985) in which LE is obtained by calculating the growth rate of area and volume in an extended phase space. However, this technique, albeit simple, is able only to find non–negative LE and generally one or two exponents alone and hence it is impossible to get a complete LE spectrum. The other scheme (Sano and Sawada, 1988) consists of correlation matrix analysis, with which an entire LE spectrum can be obtained from a 1D time series. Eckmann(1986) reported encouraging result in his investigation of a complete LE spectrum of the Lorenz attractor by use this procedure. Here, we propose an entire LE spectrum based on the 1970—1985 series of daily mean pressure of Shanghai for the behavior of a weather attractor in phase space.

II. EXTRACTION OF LE SPECTRUM FROM THE TIME SERIES

For a dynamic system in a d–dimensional phase space we assume the equation for its evolution to be in the form

$$\dot{\mathbf{x}} = F(\mathbf{x}) \ . \tag{1}$$

The tangential vector in the $\mathbf{x}$(t) tangential space is set to be ξ, whole evolution can be expressed by linearizing (1), which taken on the form

$$\dot{\xi} = T(\mathbf{x}(t))\xi \ , \tag{2}$$

where $T = \partial F / \partial x$ is the Jacobin matrix of F, and for the solution of (2) we may have

$$\xi(t) = A^{t}\xi(0) \ , \tag{3}$$

where A^t is the linear operator for the evolution of the tangential vector from $\xi(0)$ to $\xi(t)$.

Thus, the divergence of the mean LEs of the vector is defined as

$$\lambda(\mathbf{x}(0),\xi(0)) = \lim_{t\to\infty}\frac{1}{t}\ln\frac{|\xi(t)|}{|\xi(0)|} \ . \tag{4}$$

The vector has its modes denoted by $|\cdots|$. Assuming $\xi(0)$ to have $\{\mathbf{e}_i\}$as its base in the d–dimensional phase space, we have $\lambda_i(x(0)) = \lambda(\mathbf{x}(0),\mathbf{e}_i)$ as the evaluations of λ on this base with their order as $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_d$ that constitute a characteristic LE spectrum.

Let $\{x_i\}$(i = 1, 2, ...N) be the time series of actual measurements and $\triangle t$ be the sampling interval, i.e., $x_i = x[t_0+(i-1)\triangle t]$, whereupon a d–dimensional phase space is constructed with the aid of a continuation technique. With τ as time lag, we have a new sequence as follows:

$$\begin{cases} x(t_0),\ x(t_1),\ x(t_2),\ \cdots\ x(t_N) \\ x(t_0+\tau),\ x(t_1+\tau),\ x(t_2+\tau),\ \cdots\ x(t_N+\tau) \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ x(t_0+(d_E-1)\tau),\ x(t_1+(d_E-1)\tau),\ \cdots\ x(t_N+(d_E-1)\tau) \end{cases} \tag{5}$$

where each of the columns represents a point in the d_E–dimensional space and all the points constitute an orbit in the phase space. With a point $\mathbf{x}_i$ on the orbit as the center and r as the radius for a ball we have the set of points inside it in the form $S_i = \{x_j\}(j = 1,2,3,..., n)$, leading to

$$|\mathbf{x}_j - \mathbf{x}_i| \leqslant r. \tag{6}$$

For convenience,

$$x_j - x_i = \max_{0\leqslant\alpha\leqslant d_E-1} |x_{j+\alpha} - x_{i+\alpha}| \quad (\alpha = 1,2,\cdots,d_E - 1) \tag{7}$$

is employed in place of the distance in the Euclidean space. The set of points is obviously related to the evaluated r.

After a unit time elapsed when $\mathbf{x}_i$ becomes $\mathbf{x}_{i+1}$, the related $\{x_j\}$ is evolved into $\{\mathbf{x}_{j+1}\}$. Since LEs are used to investigate the averaged expansion / contraction rate with respect to a small deviation initially for the orbit, it is possible to create a linear operator to describe the time–varying deviation. For a d_E–order matrix A_i the relation of the initial distance $\mathbf{x}_j - \mathbf{x}_i$ to the distance of $\mathbf{x}_{j+1} - \mathbf{x}_{i+1}$ after a unit time is assumed to satisfy

$$\mathbf{x}_{j+1} - \mathbf{x}_{i+1} \approx A_i(\mathbf{x}_j - \mathbf{x}_i), \tag{8}$$

where A_i is the approximate value of A' in (3).

$\mathbf{x}_j - \mathbf{x}_i$ does not necessarily occupy the whole of the R^d space. For instance, a 4–D phase space holds 3–D Lorenz system. For this reason, it is possible to reduce the size of dimensionality of A_i and we assume an integer m $(m>1)$ to satisfy

$$d_E - 1 = m(d_M - 1), \tag{9}$$

where $d_M < d_E$ is available with d_M as the matrix dimension.

With such transformation, corresponding to the vector $\mathbf{x}_i$ for the d_E dimension in the form $x_i, x_{i+1}, \ldots, x_{i+d_E-1}$, the vector for the d_M dimension takes the form

$$Y_i = (x_i, x_{i+m}, \ldots, x_{i+(d_M-1)m}) \tag{10}$$

and the form of the vector relative to (8) for the d_E–dimension has the expression

$$\mathbf{y}_{j+1} - \mathbf{y}_{i+1} \doteq \mathbf{A}_i(\mathbf{y}_j - \mathbf{y}_0)$$

for the d_M–dimension, which is obviously equivalent to

$$\mathbf{x}_{j+m} - \mathbf{x}_{i+m} \doteq \mathbf{A}_i(\mathbf{x}_j - \mathbf{x}_i). \tag{11}$$

It is worth noting that, based on (10) and (11), A_i should be in the form

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \ldots & \ldots & a_{d_M} \end{bmatrix}, \tag{12}$$

where a_k is obtained in virtue of the following expression by means of the least squares fit, viz.,

$$\sum_{j\in S_i} \left\{ \sum_{k=0}^{d_M-1} a_{k+1}(x_{j+km} - x_{i+km}) - (x_{j+d_Mm} - x_{i+d_Mm}) \right\}^2 = \min. \tag{13}$$

From (12)—(13) a series of matrices can be obtained as A_i, A_{i+m}, $A_{i+2m}\cdots$, which are now renumbered as A_1, A_{1+m}, $A_{1+2m}\cdots$.

We assume $Q_{(j)}$ to be an orthogonal matrix in which $Q_{(0)}$ is a unit matrix and $R_{(j)}$ an upper triangle matrix with positive elements on the diagonal. With the orthogonal decomposition of Schmi(Luan, 1984), we have

$$\begin{cases} A_1 Q_{(0)} = Q_{(1)} R_{(1)} , \\ A_{1+m} Q_{(1)} = Q_{(2)} R_{(2)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots , \\ \cdots\cdots\cdots\cdots\cdots\cdots , \\ A_{1+jm} Q_{(j)} = Q_{(j+1)} R_{(j+1)} . \end{cases} \tag{14}$$

Thus LE λ_k takes the form

$$\lambda_k = \frac{1}{m \triangle t K} \sum_{j=1}^{K} \ln R_{(j),kk} , \tag{15}$$

where K is the number of the $R_{(j)}$ matrices with $K=(N-d_M m-1)/m$, $\triangle t$ the sampling interval, m the integer values in (9) and $R_{(j),\,kk}$ the elements on the major diagonal of the matrix $R_{(j)}$.

III. CALCULATIONS

The sample used in this work is taken from the Data Compilation published by Central Meteorological Bureau, PRC(1970—1985) and has the length $N=5000$ and $\triangle t=1$ day.

As noted in Yan et al.(1991), with time lag $\tau=5$, the extended phase space has its coordinate components independent of each other so that the system's dynamic characteristic values obtained are stable, indicating that the related dimensionality ranges from 7.7 to 7.9 for the series. Therefore, a proper d_M is taken over the range of 6, 7, ..., 13 in the present work.

It is important to evaluate r in (6). Thus, we adjust r in a sphere $\{r_{\min} < |\mathbf{x}_j - \mathbf{x}_i| < r\}$ so that any sphere centered at $\mathbf{x}_i$ contains the number of neighboring points N in the range $20 < N < 50$. m is a positive integer taken as 2, 3, and 4 in calculation.

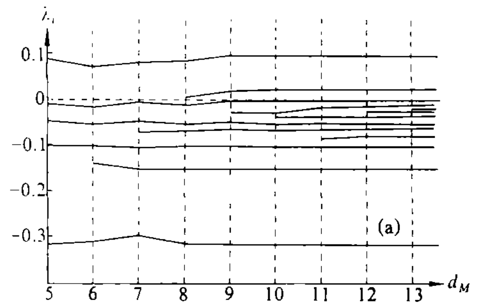
Before computation we perform smoothing of the raw data to reduce the noise. The data series treated that way is put into calculation, separately, in terms of (5) through (15), with the results illustrated below.

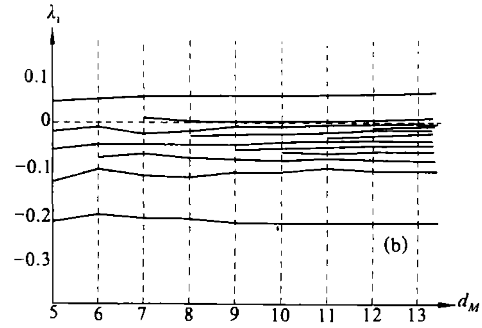
(1) For the phase space extended by $\tau=5$ the matrix dimensionality d_M bears the relation to the Lyapunov exponent λ_i with different m, as shown in Fig.1.

Table 1. LE Spectrum with $\tau=5$, $m=2$ and $d_M=9$

λ_i	LE magnitude	Sign	λ_i	LE magnitude	Sign
λ_1	0.096305	+	λ_6	−0.067100	−
λ_2	0.014100	+	λ_7	−0.095120	−
λ_3	−0.000195	0	λ_8	−0.165408	−
λ_4	−0.034436	−	λ_9	−0.340828	−
λ_5	−0.053214	−			

$\sum \lambda_i = -0.645701 < 0$ $\quad \sum_{\lambda_i > 0} \lambda_i = 0.110405$





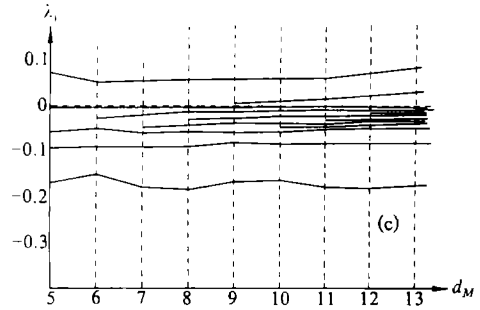


Fig.1. Relation between d_M and λ_i with $\tau=5$ for $m=2$ (a), $m=3$ (b) and $m=4$ (c).

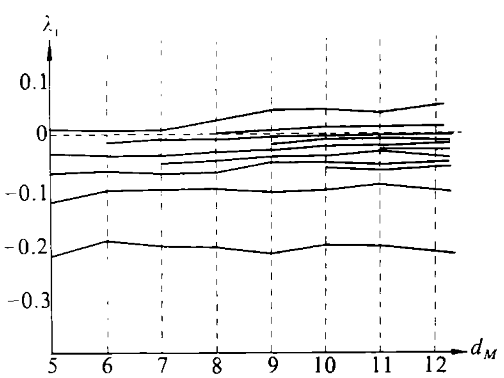


Fig.2. Relation between d_M and λ_i for $\tau=3$ and $m=3$.

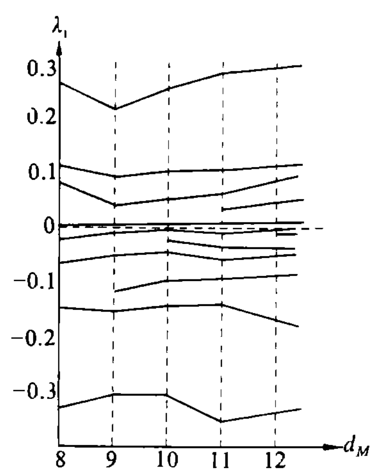


Fig.3. Relation between d_M and λ_i with $\tau=5$, $m=2$ and $5<N<15$.

It is clear from Fig.1 that for the different d_M, the number of λ_i varies, and from Fig.1a that the LEs' values become steady after $d_M=9$. Table 1 presents the LEs for $\tau=5$, $m=2$ and $d_M=9$.

(2) As portrayed in Fig.2, the LE spectrum has unstable nature in the phase space extended by $\tau=3$, which is caused, as indicated in Wolf(1985), by the dependence of the coordinate components on each other in such a space and thus the characteristic values obtained of the dynamic system are variable.

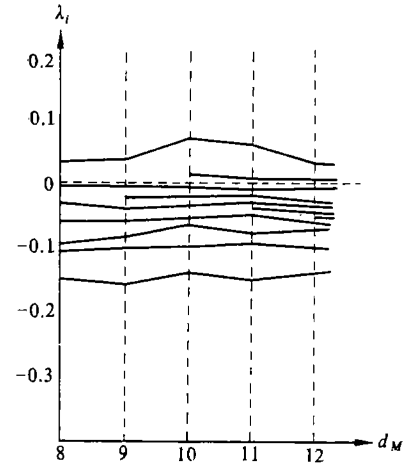


Fig.4. Relation of d_M to λ_i with $\tau=5$, $m=2$ and noise retained.

(3) If the sphere centered at $\mathbf{x}_i$ contains too few neighboring points to meet the needs of statistical averaging, then the LEs are variable in magnitude. Fig.3 delineates the number of neighboring points $5<N<15$ inside a sphere for $\tau=5$ and $m=2$.

(4) Fig.4 illustrates the LE spectrum for $\tau=5$ and $m=2$ based on the raw data (unsmoothed).

Comparison of Fig.4 to Fig.1a indicates that the smoothing is of help because the noise in the series can give rise to the instability of the LEs′ values.

IV. CONCLUSIONS

In view of the fact that Shanghai and its surroundings are marked climatically by tens of cyclones attending, control of the subtropical high lasting 30—50 days in summer and autumn and 3 to 5 processes related to tropical cyclones each year, we make use of the daily mean pressure dataset over 1970—1985 there, during which were recorded 3 warm and 7 severe winters, and 5 cool and 4 scorching summers. For the complex weather–climate system composed of so many weather processes it is necessary to first improve our understanding of its development on the whole in the dynamic research. Evidently, the extraction of the LE spectrum out of a time series is a useful approach. From the calculations discussed above we come to the following conclusions.

(1) A complete LE spectrum in the extended phase space is presented based on the Shanghai pressure series for 1970—1985 in terms of the correlation matrix analysis method, and shows that LEs can be larger or smaller than, or equal to zero. For a dissipative system the sum of the LEs ($\sum \lambda_i$) is known to be <0, indicating that its phase volume is contractive in the evolution. The presence of positive LE in our study shows the short–term evolution of the weather–climate system to be chaotic, a result identical with that of Yan et al. (1991), which provides a new field of vision on the background of the evolution of the weather–climate system in the subtropical humid climate.

(2) The LE spectrum obtained from the phase space extended by $\tau=5$ is quite stable. As shown by Yan et al.(1991), this is the case where the coordinate components are independent of each other in the space, leading to stable characteristic values of the system. The optimum LE

spectrum is obtained with $m=2$ in the $\tau=5$ phase space. We have $d_E=2(d_M-1)$ from (9) with $m=2$. This is close to the relation of the integer for the related dimensionality to be closest to the saturation insert dimension, as shown in Table 1 of Yan et al.(1991).

(3) Table 1 gives LE values of a complete spectrum, indicating two positive, one zero and six negative values, with the sum of the positive ($\sum_{\lambda_i>0} \lambda_i$) being 0.110405 for the bodily dilatation of the system, in agreement with the Kolmogorov entropy K obtained. Therefore, their relation (Rulle, 1983) is $K \doteq \sum_{\lambda_i>0} \lambda_i = 0.110405$. The inverse of K is $T=1/K$, a measure for estimating the predictable time scale of the system, and $T \doteq 9$ days has been found out in our present study, an outcome quite close to that obtained by the dynamic–statistical technique and to the entropy directly extracted from the series used (Yan et al.,1991).

(4) LE is the quantity to describe the rate of the occurrence or disintegration of indefinite factors of the system. The initial undeterminancy depends on the maximum LE ($\lambda_{\max}$) for its rate in covering the whole attractor and so does the time when the perturbation of the attractor is on the verge of disappearance on $\lambda_{\min}$. It is clear from Fig.1 that $\lambda_{\max}$ and $\lambda_{\min}$ are kept unchanged although the number of LEs increases with growing d_M at it>9, showing that the values of our LE spectrum are reliable.

It is encouraging that we get similar results from the same sample used both in Yan et al. and this article, indicating the usefulness of the methods we apply. In the former case the related dimension and Kolmogorov entropy are found out by a direct computation while in the latter the LE spectrum is extracted directly from the series by means of the correlation matrix analysis. More important, the short–term weather evolution over the area of Shanghai is really of chaotic nature.

Fractional dimension, LE spectrum and Kolmogorov entropy are central quantities in the study of chaotic motions. A chaotic motion is shown as a strange attractor which is a set of limits of low–dimensional phase space so that a definite differential equation of a limited number of variables can be used to make a forecast over a mean period with respect to the weather system evolution of this sort. Since a chaotic motion is sensitive to initial conditions such forecasts must be confined within a certain scale of time. For those beyond the limit a procedure of statistical analysis has to be applied.

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