THE FORMULATION OF FIDELITY SCHEMES OF PHYSICAL CONSERVATION LAWS AND IMPROVEMENTS ON A TRADITIONAL SPECTRAL MODEL OF BAROCLINIC PRIMITIVE EQUATIONS FOR NUMERICAL PREDICTION*

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ABSTRACT

In this paper, two formulation theorems of time-difference fidelity schemes for general quadratic and cubic physical conservation laws are respectively constructed and proved, with earlier major conserving time-discretized schemes given as special cases. These two theorems can provide new mathematical basis for solving basic formulation problems of more types of conservative time-discrete fidelity schemes, and even for formulating conservative temporal-spatial discrete fidelity schemes by combining existing instantly conserving space-discretized schemes. Besides, the two theorems can also solve two large categories of problems about linear and nonlinear computational instability.

The traditional global spectral-vertical finite-difference semi-implicit model for baroclinic primitive equations is currently used in many countries in the world for operational weather forecast and numerical simulations of general circulation. The present work, however, based on Theorem 2 formulated in this paper, develops and realizes a high-order total energy conserving semi-implicit time-difference fidelity scheme for global spectral-vertical finite-difference model of baroclinic primitive equations. Prior to this, such a basic formulation problem remains unsolved for long, whether in terms of theory or practice. The total energy conserving semi-implicit scheme formulated here is applicable to real data long-term numerical integration.

The experiment of thirteen FGGE data 30-day numerical integration indicates that the new type of total energy conserving semi-implicit fidelity scheme can surely modify the systematic deviation of energy and mass conserving of the traditional scheme. It should be particularly noted that, under the experiment conditions of the present work, the systematic errors induced by the violation of physical laws of conservation in the time-discretized process regarding the traditional scheme designs (called type Z errors for short) can contribute up to one-third of the total systematic root-mean-square (RMS) error at the end of second week of the integration and exceed one half of the total amount four weeks afterwards. In contrast, by realizing a total energy conserving semi-implicit fidelity scheme and thereby eliminating corresponding type Z errors, roughly an average of one-fourth of the RMS errors in the traditional forecast cases can be reduced at the end of second week of the integration, and averagely more than one-third reduced at integral time of four weeks afterwards. In addition, experiment results also reveal that, in a sense, the effects of type Z errors are no less great than that of the real topographic forcing of the model. The

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prospects of the new type of total energy conserving fidelity schemes are very encouraging.

Key words: global spectral model for baroclinic primitive equations, total energy conserving semi-implicit fidelity scheme, type Z systematic errors, physical conservation laws, medium-range numerical prediction

I. INTRODUCTION

Physical laws of conservation of energy, mass and so on are fundamental laws in natural world. It is a basic requirement of a well-reasoned and reliable discretized computation to retain the characteristics of physical conservation laws of its original continuous system. The satisfying of such a requirement is still more important for long-term numerical integration. Apparently, even though the sources or sinks of corresponding systematic errors are not large or exceedingly small in terms of quantity, their long-term negative accumulative effects can not be overlooked. Except in special cases nonlinear problems have no analytical solutions. We can only derive solution by way of discrete numerical computations. Consequently, the basic problems of this type are normally unavoidable.

Computational stability and convergence are essential questions of numerical computations. For linear (rized) modeling, computational stability and convergence can be ensured by setting up a corresponding compatible discretized computational scheme that satisfies universal linear computational stability criterion. But for nonlinear modeling, general criteria or methods guaranteeing nonlinear computational stability are still lacking. Nor is there any equivalent theorem of stability and convergence available. How to formulate a scheme capable of reasonably preventing nonlinear computational instability and ensuring nonlinear convergence remains an essential theoretical question at which researchers of computational mathematics and physics, including those in other numerical computational fields, have long worked. And yet, for evolution problem involving some types of physical conservation laws, some square or weight square conservation laws for instance, formulating a discretized scheme capable of retaining corresponding conserving integral property itself would also mean the solving of its problems of linear and nonlinear computational instability.

Studies on the formulation of conserving schemes can be traced back to early this century (Galerkin et al. 1915). Although some instantly conserving schemes (ensuring conservation in space only and not in time) have been successfully formulated in the world (Arakawa 1966; Arakawa and Lamb 1977; Arakawa and Lamb 1981; Bryan 1966; Corby et al. 1972; Grammeltvedt 1969; Grimmer and Shaw 1967; Lilly 1965; Sadourny 1972; Simmons and Burridge 1981; Williamson 1970), yet, because of various reasons, except in the trivial first-order conserving schemes and some other very special cases, the basic formulation problem of conserving schemes in temporal-spatial discrete sense remains unsolved for long. The toughest aspect has been the construction of time discretized conserving schemes. In the recent past, with regard to a certain special operator equation, (instantly linearized) implicit and explicit time-difference scheme preserving a specified square conserving property have been successfully constructed (Zeng 1979; Wang and Ji

1990). The applying of such schemes has also resulted in the formulation and actualization of square energy conserving temporal-spatial difference (instantly linearized) implicit and explicit scheme, and a square energy conserving space-Galerkin approximation (instantly linearized) implicit time-difference scheme for barotropic primitive equations respectively under a specified transformed expression (Wang and Ji 1990; Zeng and Zhang 1982; Zhang 1982). But it is a pity that such schemes have very rigid applicable conditions. They can not be employed directly to formulate energy conserving schemes for barotropic or baroclinic primitive equations in traditional meteorological forms. In fact, it is also impossible to satisfy the stringent applicable conditions and settle the formulation problem of high-order total energy conserving time discrete schemes by way of transforming baroclinic primitive equations. Besides, the equivalent transformed expression of an equation in continuous sense generally does not possess equivalence after being discretized.

Recently the author of this paper has formulated and proved two theorems for square conserving semi-implicit time-difference schemes (Zhong 1992a; Zhong 1992b). Because of the special designs involved, the theorems can be employed in the case of semi-implicit, explicit, and (instantly linearized) implicit time-difference schemes as well as in solving formulation problems of the specified square conservative property regarding the special operator equation mentioned above and non-specified square conservative properties of non-special operator equations. The new formulation theorems (Zhong 1992c; Zhong 1993) for weight square conserving scheme and square conserving scheme formulated and proved by the author at a later time can set the former theorems (Wang and Ji 1990; Zhong 1992a; Zhong 1992b) as their special cases. The applying of these theorems to corresponding meteorological models has also led to the formulation and actualization of time difference-spatial spectral expansion enstrophy square conserving explicit scheme and square kinetic conserving explicit scheme for barotropic vorticity equations which respectively belongs and does not belong to the specified square conservative property of the special operator equations (Zhong 1992b; Zhong 1995a; Zhong 1995b). Also formulated and actualized is a time difference-spatial spectral expansion weight square energy conserving semi-implicit scheme for barotropic primitive equations (Zhong 1995a). Based on the general principles and methods established in the shaping of the above-mentioned theorems (Zhong 1992c; Zhong 1993), a time difference-spatial spectral expansion enstrophy and angular momentum conserving semi-implicit scheme for barotropic primitive equations have been respectively formulated and realized (Zhong 1993; Zhong 1995b). These theorems (Zhong 1992a; Zhong 1992b; Zhong 1992c; Zhong 1993), however, are not workable either for solving the formulation problem of high-order total energy conserving time difference scheme for baroclinic primitive equations. A basic aim of the present work is to resolve two extensive types of formulation problems pertinent to high-order conserving time difference fidelity schemes and in particular provide appropriate formulation theorem for constructing high-order total energy conserving (semi-implicit) time-difference scheme for baroclinic primitive equations.

Numerical weather predictions are, without doubt, of tremendous social and economic value. Ever since 1950s when the first and earliest successful numerical weather forecasting was produced in the world (Charney et al. 1950), numerical prediction models have undergone dramatic improvements from the relatively simple barotropic vorticity equation model at the initial stage to the complicated baroclinic primitive equations model widely used today which approximates more closely the real properties of the atmosphere; from the initial grid point model in which spatial derivatives possess poor discretized precision to the widely used spectral model in which spatial derivatives can be discretized with high precision; from earlier explicit time-difference schemes of which the discrete computational stability requirements are high but computational efficiency is low to the semi-implicit time-difference schemes widely used today of which the discrete computational stability requirements are low and yet computational efficiency is high. Whatever complexities they involve, all the forecasting models ever used for numerical weather prediction unexceptionably contain high-order physical conservation laws, hence, throughout each developmental stage of numerical weather prediction there have been remarkable efforts devoted to the formulating of corresponding conserving schemes. Nevertheless, as far as baroclinic primitive equations are concerned, although high-order total energy conserving vertical finite-difference schemes have been successfully constructed in the world for long, the basic formulation problem of high-order total energy conserving time-difference (discrete) schemes remains unsolved in both theory and practice. Thus, another primary goal of this paper is to formulate and actualize a high-order total energy conserving time-difference scheme for baroclinic primitive equations, to test the feasibility of the total energy conserving semi-implicit scheme for global spectral model by using FGGE weather data (FGGE: First GARP Global Experiment, 1 Dec. 1978 — 30 Nov. 1979. GARP: Global Atmospheric Research Program), and explore its potentials for applications.

In the following Section II of this paper, two formulation theorems of time-difference fidelity schemes, capable of retaining the general quadratic and cubic physical conservation law, will be formulated and proved. A total energy conserving (semi-implicit) time-difference scheme for global spectral model of baroclinic primitive equations will be presented in Section III. In Section IV, by using FGGE data, comparison experiment of thirteen 30-day numerical integrations between a traditional and a new total energy conserving semi-implicit time-difference scheme for global spectral model of baroclinic primitive equations will be designed and conducted. Finally, in Section V a summary of the present work is given.

II. FORMULATION PRINCIPLE AND THEOREMS FOR FIDELITY SCHEME OF HIGH-ORDER PHYSICAL CONSERVATION LAWS

The operator equation of evolution problem can be

$$\frac{\partial u}{\partial t} + Au = 0. \tag{1}$$

Based on a general compensation principle and inverse formulation method of a fidelity scheme (Zhong 1992c; Zhong 1993) that maintains the single or multiple characteristics of the original continuous system by accordingly and averagely compensating and eliminating the discrete computational errors at each computational component (grid) in accordance with the source and manner of the introduction of the errors, a general physical conservation law time-difference fidelity scheme of equation (1) can be written as

$$\frac{u^{n+1}-u^n}{\Delta t} + (A^n - A_L^n)u^n + A_L^n u^{n+1} + \Delta t \varepsilon^n B^n u^n = 0. \tag{2}$$

Here, if the auxiliary formulation operator A_L is respectively set as a linear (or linearized) part of operator A, zero and A itself then, apparently, scheme (2) is a semi-implicit, explicit or (instantly linearized) implicit scheme respectively; ε^n is an undetermined compensation coefficient with single (or multiple) component (s); compensation operator $B^n u^n$ can be given as

$$B^n u^n = -\sum_{j=1}^{J} \Delta t^{j-1}\left[\frac{1}{(j+1)!}\frac{\partial^{j+1} u^n}{\partial^{j+1} t} + \varepsilon_0 \frac{A_L^n}{j!}\frac{\partial^j u^n}{\partial^j t}\right], \quad J \text{ is positive integer} \tag{3}$$

where ε_0, as a switch constant, can be set as zero or 1; the derivative of u at any order can be determined by Eq. (1). In actual computations, the compensation operator $B^n u^n$ can be determined by Eq. (3) or its approximations. An appropriate selection of compensation operator B is very important. The fidelity scheme is universally valid for all the semi-implicit, explicit or (instantly linearized) implicit algorithm because of the introduction of the auxiliary formulation operator A_L (Zhong 1992c; Zhong 1993).

In particular if operator equation (1) yields general quadratic conserving integral property

$$\int (A_1 u \cdot A_2 A u + A_2 u \cdot A_1 A u)\, d\sigma = 0, \tag{4}$$

or general cubic conserving integral property

$$\int (A_1 u \cdot A_2 u \cdot A_3 A u + A_1 A u \cdot A_2 u \cdot A_3 u + A_1 u \cdot A_2 A u \cdot A_3 u)\, d\sigma = 0, \tag{5}$$

the following theorems are true. Here, A_1, A_2 and A_3 are all bounded space operators independent of u and t; $d\sigma$ is a space integral element.

Theorem 1: Suppose the compensation coefficient ε^n satisfies

$$\varepsilon^n = \frac{2C_1}{1+\sqrt{1-4\Delta t^2 C_1 C_2}}, \tag{6}$$

for any n time, then Scheme (2) is a fidelity scheme with quadratic conserving integral property and is compatible to Eq. (1), where

$$C_1 = C\int (A_1 A^n u^n \cdot A_2 A^n u^n + A_1 L u^n \cdot A_2 M u^n + A_2 L u^n \cdot A_1 M u^n + \Delta t^2 A_1 M u^n \cdot A_2 M u^n)\, d\sigma,$$

$$C_2 = C\int (A_1 K B^n u^n \cdot A_2 M_1 B^n u^n)\, d\sigma,$$

K, L and M are inverse operator of $I + \Delta t A_L^n$, $I - \Delta t A_n$ and $K A_L^n A^n$ respectively; I is a unit operator; C is a reciprocal of $\int [A_1 (L + \Delta t^2 M) u^n \cdot A_2 K B^n u^n + A_2 (L + \Delta t^2 M) u^n \cdot A_1 K B^n u^n] d\sigma$.

Proof: Using Property (4), it can be easily testified that if ε^n satisfies Eq. (6) then

$$\int (A_1 u^{n+1} \cdot A_2 u^{n+1})\, d\sigma = \int (A_1 u^n \cdot A_2 u^n)\, d\sigma \tag{7a}$$

is satisfied at any n time, that is, Eq. (2) is now a fidelity scheme with quadratic conserving integral property. And apparently, from Eq. (6), we can also derive the following form

$$\lim_{\Delta t \to 0} \varepsilon^n = O(\Delta t^0). \tag{7b}$$

This implies that Eq. (2) is for the moment also compatible to Eq. (1). Q. E. D.

As a special case of Theorem 1. let A_1 and A_2 be identical, the fidelity scheme with quadratic conserving property will retrograde to a square conserving scheme (Zhong 1992c; Zhong 1993). A square conserving scheme is clearly a stable scheme. More generally, it is evident that any quadratic conserving fidelity scheme with operator A_1, A_2 and A_3 ensuring $A_1u \cdot A_2u = |A_1u \cdot A_2u|$ at any point of the entire integral space is a stable scheme also. The sufficient and necessary condition for their computational stability is

$$1 \geqslant 4\Delta t^2 C_1 C_2. \tag{8}$$

As an even more special case, let us further set A_1 and A_2 as unit operators, A_L as zero operator, then the general square conserving scheme will retrograde to a square conserving explicit time-difference scheme (Wang and Ji 1990) regarding a certain special operator A · satisfying an integral constraint

$$[u(X,t), A(u^*, X, t), X, t) u(X,t)] = 0, \tag{9}$$

and a certain special square conserving property

$$\| u^{n+1} \|^2 = \| u^n \|^2, \text{ for any } n \text{ time}. \tag{10}$$

For still special cases with integral conserving property (10), that is, cases in which not only Eq. (9) but also

$$(u(X,t), A(u^*, X, t) u(X,t)) = 0, \tag{11a}$$

are satisfied, we simply let $\varepsilon^n = 0$ and $A_L = \frac{1}{2}A$, and it follows that Eq. (2) will retrograde to a natural instantly linearized square conserving implicit scheme as follows

$$\frac{u^{n+1} - u^n}{\Delta t} + A(u^n, X, t) \frac{u^{n+1} + u^n}{2} = 0. \tag{11b}$$

If we further set $A(u^n, X, t)$ as $A(u^*, X, t)$, then a general instant linearized square conserving implicit scheme (Zeng 1979; Zeng and Zhang 1982; Zhang 1982) can be obtained as

$$\frac{u^{n+1} - u^n}{\Delta t} + A(u^*, X, t) \frac{u^{n+1} + u^n}{2} = 0, \tag{11c}$$

where X is spatial coordinate, t denotes time coordinate, and u^* is an arbitrary reference value of u. A more reasonable choice here is to set $u^* = \theta u^n + (1 - \theta) u^{n+1}$, where θ is a weighting factor. In actual computation, u^{n+1} can be substituted by its estimated approximation.

It may be noted that, Scheme (11c) has very stringent applicable conditions. In fact, it is a double square conserving scheme. Namely, Scheme (11c) yields conservation of $\| u^n \|^2$ and $\| A(u^*, X, t) \|^2$ for any n time. This implies that, for physical problems that merely satisfy one of the above-described square conserving properties, it is almost impossible to obtain desirable effects by using Scheme (11c). Apparently, under this condition, because of the presence of a false computational conservation constrains, it is equivalent to the introduction of a false computational systematic error source or sink into the scheme.

Substitute the variable u^n in operator A of Scheme (11b) with $\frac{1}{2}(u^{n+1} + u^n)$, the scheme becomes a complete implicit square conserving scheme known as Crank-Nicholson Scheme.

Theorem 2: Suppose the compensation coefficient ε^n satisfies

$$\Delta t^4 \gamma_1 (\varepsilon^n)^3 + \Delta t^2 \gamma_2 (\varepsilon^n)^2 + \gamma_3 \varepsilon^n + \gamma_4 = 0, \tag{12}$$

and its order of magnitude is $O(\Delta t^0)$, then Scheme (2) is a fidelity scheme with a cubic conserving integral property and is compatible to Eq. (1), where

$$\gamma_1 = \int A_1 LB^n u^n \cdot A_2 LB^n u^n \cdot A_3 LB^n u^n \mathrm{d}\sigma, \tag{13a}$$

$$\gamma_2 = -\int (A_1 LB^n u^n \cdot A_2 LB^n u^n \cdot A_3 M u^n + A_1 LB^n u^n \cdot A_2 M u^n \cdot A_3 LB^n u^n + A_1 M u^n \cdot A_2 LB^n u^n \cdot A_3 LB^n u^n) \mathrm{d}\sigma, \tag{13b}$$

$$\gamma_3 = \int (A_1 LB^n u^n \cdot A_2 M u^n \cdot A_3 M u^n + A_1 M u^n \cdot A_2 LB^n u^n \cdot A_3 M u^n + A_1 M u^n \cdot A_2 M u^n \cdot A_3 LB^n u^n) \mathrm{d}\sigma, \tag{13c}$$

$$\gamma_4 = -\int (A_1 K u^n \cdot A_2 A^n u^n \cdot A_3 A^n u^n + A_1 A^n u^n \cdot A_2 u^n \cdot A_3 A^n u^n + A_1 A^n u^n \cdot A_2 A^n u^n \cdot A_3 u^n + A_1 LA_L^n A^n u^n \cdot A_2 M u^n \cdot A_3 M u^n + A_1 K u^n \cdot A_2 M u^n \cdot A_3 LA_L^n A^n u^n + A_1 K u^n \cdot A_2 LA_L^n A^n u^n \cdot A_3 K u^n) \mathrm{d}\sigma, \tag{13d}$$

$K = I - \Delta t A^n$, $M = I - \Delta t A^n + \Delta t^2 LA_L^n A^n$, L is an inverse operator of $I + \Delta t A_L^n$, I is a unit operator.

Proof: Using Property (5), it is easy to prove that if ε^n satisfies Eq. (12), then

$$\int (A_1 u^{n+1} \cdot A_2 u^{n+1} \cdot A_3 u^{n+1}) \mathrm{d}\sigma = \int (A_1 u^n \cdot A_2 u^n \cdot A_3 u^n) \mathrm{d}\sigma \tag{14}$$

is true at any n time, that is, Eq. (2) is now a fidelity scheme with cubic conserving integral property. Apparently, in accordance with Eq. (12), we can derive the following expression

$$\lim_{\Delta t \to 0} \varepsilon^n = \lim_{\Delta t \to 0} (-\gamma_4 / \gamma_3) = O(\Delta t^0). \tag{15}$$

This implies that Eq. (2) is now also compatible to Eq. (1). Q. E. D.

As a special case of Theorem 2, set A_1 as a bounded positive definite operator, A_2 and A_3 identical bounded operators and all are independent of u and t, then the cubic conserving fidelity scheme is further degenerated into a weight square conserving fidelity scheme (Zhong 1992b; Zhong 1993). A fidelity scheme with weighted square conserving integral property is, clearly, an absolutely stable computational scheme.

More generally, it is apparent that any cubic conserving fidelity scheme with operators A_1, A_2 and A_3 ensuring $A_1 u \cdot A_2 u \cdot A_3 u = |A_1 u \cdot A_2 u \cdot A_3 u|$ at any point of the entire integral space is an absolutely stable computational scheme.

III. THE FORMULATION OF A TOTAL ENERGY CONSERVING SEMI-IMPLICIT TIME-DIFFERENCE SCHEME FOR GLOBAL SPECTRAL-VERTICAL FINITE-DIFFERENCE MODEL OF BAROCLINIC PRIMITIVE EQUATIONS

1. *Model Control Equation (s)*

Set vertical coordinate (Phillips 1957) $\sigma = p/p_s$, where p is pressure, p_s surface pressure. Suppose change of water vapor in model atmosphere is neglected, model atmosphere is free of friction and free of heat exchange with the outside world, then the vorticity, divergence, thermodynamic, continuity and hydrostatic equation under the horizontal spherical coordinate system may be respectively written as

$$\frac{\partial \zeta}{\partial t} - \frac{1}{a(1-\mu^2)}\frac{\partial Fv}{\partial \lambda} + \frac{1}{a}\frac{\partial Fu}{\partial \mu} = 0, \tag{16}$$

$$\frac{\partial D}{\partial t} - \frac{1}{a(1-\mu^2)}\frac{\partial Fu}{\partial \lambda} - \frac{1}{a}\frac{\partial Fv}{\partial \mu} + \nabla^2\left[\frac{U^2+V^2}{2(1-\mu^2)} + \Phi\right] = 0, \tag{17}$$

$$\frac{\partial T}{\partial t} + \frac{U}{a(1-\mu^2)}\frac{\partial T}{\partial \lambda} + \frac{V}{a}\frac{\partial T}{\partial \mu} + \dot{\sigma}\frac{\partial T}{\partial \sigma} - \frac{RT}{c_p}\cdot\frac{\omega}{p} = 0, \tag{18}$$

$$\frac{\partial p_s}{\partial t} + \nabla\cdot(v_h p_s) + \frac{\partial p_s\dot{\sigma}}{\partial \sigma} = 0, \tag{19}$$

$$\frac{\partial \varphi}{\partial \ln\sigma} + RT = 0 \quad \text{or} \quad \Phi = \frac{\partial \Phi\sigma}{\partial \sigma} + RT. \tag{20}$$

Using model atmosphere vertical bounded condition $\dot{\sigma} = 0$ when $\sigma=0$ and $\sigma=1$, integrate Eq. (14), we can get

$$\frac{\partial p_s}{\partial t} + \int_0^1 \nabla\cdot(v_h p_s)\,d\sigma = 0 \quad \text{or} \quad \frac{\partial \ln p_s}{\partial t} + \int_0^1 (D + v_h\cdot\nabla\ln p_s)\,d\sigma = 0, \tag{21}$$

$$\begin{aligned}\dot{\sigma} &= \sigma\int_0^1 \left(D + \frac{v_h}{p_s}\cdot\nabla p_s\right) - \int_0^\sigma \left(D + \frac{v_h}{p_s}\cdot\nabla p_s\right)d\sigma \\ &= \sigma\int_0^1 (D + v_h\cdot\nabla\ln p_s) - \int_0^\sigma (D + v_h\cdot\nabla\ln p_s)\,d\sigma,\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\omega}{p} &= \frac{1}{p_s}v_h\cdot\nabla p_s - \frac{d\ln\sigma}{d\sigma}\int_0^\sigma \left(D + \frac{v_h}{p_s}\cdot\nabla p_s\right)d\sigma \\ &= v_h\cdot\nabla\ln p_s - \frac{d\ln\sigma}{d\sigma}\int_0^\sigma (D + v_h\cdot\nabla\ln p_s)\,d\sigma,\end{aligned} \tag{23}$$

where t denotes time, λ, θ are respectively earth longitude and latitude; $\mu=\sin\theta$, u, v are longitudinal, meridional components of horizontal wind vector v_h respectively; u is positive towards east, v is positive towards north; T is absolute temperature, $\zeta=\nabla^2\Psi$ represents vertical component of relative vorticity, $D=\nabla^2\chi$ is horizontal divergence, $\varphi(=gz)$ geopotential height, $\dot{\sigma}=\frac{d\sigma}{dt}$ is σ-coordinate vertical velocity, $\dot{\omega}=\frac{dp}{dt}$ is p-coordinate vertical velocity, Ψ stream function, χ is velocity potential, z is height, ∇ denotes horizontal gradient operator, ∇^2 denotes horizontal Laplace operator. Furthermore,

$$Fu = V(f+\zeta) - \dot{\sigma}\frac{\partial U}{\partial \sigma} - \frac{RT}{a}\frac{\partial \ln p_s}{\partial \lambda} = V(\zeta+f) - \dot{\sigma}\frac{\partial U}{\partial \sigma} - \frac{RT}{p_s}\frac{\partial p_s}{a\partial \lambda}, \tag{24}$$

$$\begin{aligned}Fv &= -U(f+\zeta) - \dot{\sigma}\frac{\partial V}{\partial \sigma} - \frac{RT}{a}(1-\mu^2)\frac{\partial \ln p_s}{\partial \mu} \\ &= -U(\zeta+f) - \dot{\sigma}\frac{\partial V}{\partial \sigma} - \frac{RT}{p_s}(1-\mu^2)\frac{\partial p_s}{a\partial \mu},\end{aligned} \tag{25}$$

$$U = u\cos\theta = \frac{1}{a}\left[-(1-\mu^2)\frac{\partial \Psi}{\partial \mu} + \frac{\partial \chi}{\partial \lambda}\right], \tag{26}$$

$$V = v\cos\theta = \frac{1}{a}\left[\frac{\partial \Psi}{\partial \lambda} + (1-\mu^2)\frac{\partial \chi}{\partial \mu}\right], \tag{27}$$

U, V are scaled zonal and meridional winds respectively, $f=2\Omega\sin\theta$ is Coriolis parameter, Ω is angular velocity of earth, a is radius of earth, R and c_p are respectively air constant and specific heat of dry air at constant pressure. It can be verified that the motions described by baroclinic primitive equations (16—20) constantly possess cubic high-order total energy conserving global integral property

$$\iint\limits_S\int_0^1 p_s[\Phi_S + c_pT + \frac{U^2+V^2}{2(1-\mu^2)}]\mathrm{d}\sigma\mathrm{d}s = \text{constant}. \tag{28}$$

Here, Φ_S is surface geopotential height, ds is spherical integral element, S total area of integral sphere.

2. *The Traditional Semi-Implicit Scheme for Global Spectral-Vertical Finite-Difference Model*

With ζ, D, T, $\ln p_s$ as variable and Φ as diagnostic variable, using semi-implicit time integration scheme (Robert et al. 1972) for divergence, temperature and surface pressure equation and introducing time filtering (Asselin 1972) to prevent the false growth of computational mode, the traditional semi-implicit scheme of the global spectral-vertical finite-difference model for baroclinic primitive equations can be given as

$$\delta_t\zeta_{m,l,k} + ZT_{m,l,k} = 0, \tag{29}$$

$$\delta_t D_{m,l,k} + DT_{m,l,k} + \left[\nabla^2\left(\frac{U^2+V^2}{2(1-\mu^2)}\right)+\Phi\right]_{m,l,k}$$
$$+\nabla^2\left[\sum_{j=k+1}^{NL}R\Delta_u T_j\ln\frac{\sigma_{j+\frac{1}{2}}}{\sigma_{j-\frac{1}{2}}} + \alpha_k R\Delta_u T_k + RT_0\Delta_u\ln p_s\right]_{m,l} = 0, \tag{30}$$

$$\delta_t T_{m,l,k} + TT_{m,l,k} + \frac{RT_0}{c_p}\left(\frac{\ln\sigma_{k+\frac{1}{2}}/\sigma_{k-\frac{1}{2}}}{\Delta\sigma_k}\sum_{j=1}^{k-1}D_j\Delta\sigma_j + \alpha_k\Delta_u D_k\right)_{m,l} = 0, \tag{31}$$

$$\delta_t(\ln p_s)_{m,l} + PT_{m,l} + \sum_{j=1}^{NL}\Delta_u D_j\Delta\sigma_j = 0, \tag{32}$$

where $X_{m,l,k}$ is coefficient of spectral expansion with spherical harmonics $p_l^m(\mu)e^{im\lambda}$ as basis function for variable X at layer k, operator δ_t, Δ_u are respectively $\delta_t X = (X^+ - X_f^-)/2\Delta t$ and $\Delta_u X = (X^+ - X_f^-)/2 - X$, where X denotes a variable at t time, X^+ represents variable X at time $t+\Delta t$, X_f^- represents X at time $t-\Delta t$ after time filtering (Asselin 1972). X_f is obtained from $X_f = X + \varepsilon_f(X_f^- - 2X + X^+)$ after integrating (29) — (32) one step forward. Spectral coefficient $ZT_{m,l,k}$, $DT_{m,l,k}$, $TT_{m,l,k}$ at any k layer and $PT_{m,l}$ are derived directly by grid-point/spectral transform after obtaining the corresponding value of nonlinear terms ZT, DT, TT and PT at each computational grid point of three-dimension space at present time t. T_0 represents reference temperature of atmosphere. Here

$$ZT_k = -\frac{1}{a(1-\mu^2)}\frac{\partial(Fv)_k}{\partial\lambda} + \frac{\partial(Fu)_k}{a\partial\mu}, \tag{33a}$$

$$DT_k = -\frac{1}{a(1-\mu^2)}\frac{\partial(Fu)_k}{\partial\lambda} - \frac{\partial(Fv)_k}{a\partial\mu} + \nabla^2\left[\frac{U_k^2+V_k^2}{2(1-\mu^2)} + \Phi_k\right], \tag{33b}$$

$$TT_k = \frac{U_k}{a(1-\mu^2)}\frac{\partial T_k}{\partial\lambda} + \frac{V_k}{a}\frac{\partial T_k}{\partial\mu} + (\dot\sigma\frac{\partial T}{\partial\sigma})_k - \frac{RT_k}{c_p}\left(\frac{\omega}{p}\right)_k, \tag{33c}$$

$$PT = \sum_{j=1}^{NL}(v_{h_j}\cdot\nabla\ln p_s + D_j)\Delta\sigma_j, \tag{33d}$$

$$(Fu)_k = V_k(\zeta_k + f) - (\dot\sigma\frac{\partial U}{\partial\sigma})_k - RT_k\frac{\partial\ln p_s}{a\partial\lambda}, \tag{33e}$$

$$(Fv)_k = -U_k(\zeta_k + f) - (\dot{\sigma}\frac{\partial V}{\partial \sigma})_k - RT_k(1-\mu^2)\frac{\partial \ln p_s}{a\partial \mu}. \tag{33f}$$

To represent the vertical variation of all dependent variable, the atmosphere is divided into NL layers. Variables ζ, D, T are defined at intermediate integer levels (full levels), and $\dot{\sigma}$ is defined at interface levels (the "half levels") between these layers. Define $\dot{\sigma}_{\frac{1}{2}} = \dot{\sigma}_{NL+\frac{1}{2}} = 0$ and other $\dot{\sigma}_{k+\frac{1}{2}}$ at half levels as

$$\dot{\sigma}_{k+\frac{1}{2}} = \sigma_{k+\frac{1}{2}}\sum_{j=1}^{NL}(D_j + \mathbf{v}_{h_j}\cdot\nabla\ln p_s)\Delta\sigma_j - \sum_{j=1}^{k}(D_j + \mathbf{v}_{h_j}\cdot\nabla\ln p_s)\Delta\sigma_j,$$
$$k = 1,2,\cdots,NL-1, \tag{34}$$

determine Φ at full levels and half levels through discretized hydrostatic relations (20)

$$\Phi_k = \frac{\Phi_{k+\frac{1}{2}}\sigma_{k+\frac{1}{2}} - \Phi_{k-\frac{1}{2}}\sigma_{k-\frac{1}{2}}}{\Delta\sigma_k} + RT_k = \Phi_{k+\frac{1}{2}} + \alpha_k RT_k, \tag{35a}$$

$$\Phi_{k+\frac{1}{2}} = \Phi_{k-\frac{1}{2}} - RT_k\ln\frac{\sigma_{k+\frac{1}{2}}}{\sigma_{k-\frac{1}{2}}} = \Phi_S + \sum_{j=k+1}^{NL}RT_j\ln\frac{\sigma_{j+\frac{1}{2}}}{\sigma_{j-\frac{1}{2}}}, \tag{35b}$$

$$\alpha_k = 1 - \frac{\sigma_{k-\frac{1}{2}}}{\Delta\sigma_k}\ln\frac{\sigma_{k+\frac{1}{2}}}{\sigma_{k-\frac{1}{2}}}. \tag{35c}$$

take variable X vertical advection at integer layer as

$$(\dot{\sigma}\frac{\partial X}{\partial \sigma})_k = \frac{1}{2}\left(\dot{\sigma}_{k+\frac{1}{2}}\frac{X_{k+1}-X_k}{\Delta\sigma_{k+\frac{1}{2}}} + \dot{\sigma}_{k-\frac{1}{2}}\frac{X_k - X_{k-1}}{\Delta\sigma_{k-\frac{1}{2}}}\right),$$
$$X = U, V, T, \tag{36}$$

let vertical velocity ω at plane p satisfies

$$(\frac{\omega}{p})_k = \mathbf{v}_{h_k}\cdot\nabla\ln p_s - \alpha_k(D_k + \mathbf{v}_{h_k}\cdot\nabla\ln p_s) - \frac{\ln\sigma_{k+\frac{1}{2}}/\sigma_{k-\frac{1}{2}}}{\Delta\sigma_k}\sum_{j=1}^{k}(D_j + \mathbf{v}_{h_j}\cdot\nabla\ln p_s)\Delta\sigma_j, \tag{37}$$

then it can be verified that the satisfying of (35) — (37) can ensure there is no spurious energy source or sink due to vertical finite-differencing. If we further set $\sigma_{\frac{1}{2}} = 0$ thereby $\alpha_1 = 1$, then the satisfying of (35) — (37) can also ensure there is no spurious angular momentum source or sink due to vertical differencing (Simmons and Strufing 1981).

The traditional semi-implicit time-difference schemes (29) — (32) involve three time levels therefore produce a false computational mode. To prevent its false growth, an artificial smoother or time filter (Asselin 1972) has to be introduced. This is anyhow a defect. More importantly, the absence of false energy source or sink due to time-differencing can not be ensured in this scheme.

3. *The Total Energy Conserving Semi-Implicit Time-Difference Scheme of Global Spectral-Vertical Finite-Difference Model of Baroclinic Primitive Equations*

Based on the formulation principle and method of general fidelity scheme of physical conservation laws, a total energy conserving semi-implicit time-difference scheme for global spectral-vertical finite-difference model of baroclinic primitive equations (16) — (20) can be given as follows

$$\frac{\zeta^{n+1}_{m,l,k} - \zeta^n_{m,l,k}}{\Delta t} + ZT^n_{m,l,k} + \varepsilon_n(B^n\zeta^n)_{m,l,k} = 0, \tag{38}$$

$$\frac{D^{n+1}_{m,l,k} - D^n_{m,l,k}}{\Delta t} + DT^n_{m,l,k} + \left\{ \nabla^2 \left[\frac{U^2 + V^2}{2(1-\mu^2)} + \Phi \right] \right\}^n_{m,l,k}$$

$$+ \beta_0 \nabla^2 \left[\sum_{j=k+1}^{NL} R(T^{n+1}_j - T^n_j) \ln \frac{\sigma_{j+\frac{1}{2}}}{\sigma_{j-\frac{1}{2}}} + \alpha_k R(T^{n+1}_k - T^n_k) + \frac{RT_0}{p_0}(p^{n+1}_s - p^n_s) \right]_{m,l}$$

$$+ \varepsilon_n (B^n D^n)_{m,l,k} = 0, \tag{39}$$

$$\frac{T^{n+1}_{m,l,k} - T^n_{m,l,k}}{\Delta t} + TT^n_{m,l,k}$$

$$+ \beta_0 \frac{RT_0}{c_p} \left[\frac{\ln \sigma_{k+\frac{1}{2}}/\sigma_{k-\frac{1}{2}}}{\Delta \sigma_k} \sum_{j=1}^{k-1} (D^{n+1}_j - D^n_j) \Delta \sigma_j + \alpha_k (D^{n+1}_k - D^n_k) \right]_{m,l} + \varepsilon_n (B^n T^n)_{m,l,k} = 0, \tag{40}$$

$$\frac{p^{n+1}_s - p^n_s}{\Delta t} + PT^n_{m,l} + \beta_0 p_0 \Big(\sum_{j=1}^{NL} D_j \Delta \sigma_j \Big)_{m,l} + \varepsilon_n (B^n p^n_s)_{m,l} = 0. \tag{41}$$

Here, T_0, p_0 are reference temperature and surface pressure of model atmosphere respectively, β_0 is switch constant which can be set as 0 or 1, when β_0 is 0, computational schemes (38) — (41) are an explicit scheme, when β_0 is 1, it becomes a semi-implicit scheme, and

$$ZT = -\frac{1}{a(1-\mu^2)} \frac{\partial Fv}{\partial \lambda} + \frac{\partial Fu}{a \partial \mu}, \tag{42a}$$

$$DT = -\frac{1}{a(1-\mu^2)} \frac{\partial Fu}{\partial \lambda} - \frac{1}{a} \frac{\partial Fv}{\partial \mu} + \nabla^2 \left[\frac{U^2 + V^2}{2(1-\mu^2)} + \Phi \right], \tag{42b}$$

$$TT = \frac{U}{a(1-\mu^2)} \frac{\partial T}{\partial \lambda} + \frac{V}{a} \frac{\partial T}{\partial \mu} + \dot{\sigma} \frac{\partial T}{\partial \sigma} - \frac{RT_k}{c_p} \cdot \frac{\omega}{p}, \tag{42c}$$

$$PT = \sum_{j=1}^{NL} (\mathbf{v}_{h_j} \cdot \nabla p_s + p_s D_j) \Delta \sigma_j, \tag{42d}$$

$$Fu = V(\zeta + f) - \dot{\sigma} \frac{\partial U}{\partial \sigma} - \frac{RT}{p_s} \frac{\partial p_s}{a \partial \lambda}, \tag{43a}$$

$$Fv = -U(\zeta + f) - \dot{\sigma} \frac{\partial V}{\partial \sigma} - \frac{RT}{p_s}(1-\mu^2) \frac{\partial p_s}{a \partial \mu}. \tag{43b}$$

Model atmosphere is vertically divided into NL layers. Basic variables ζ, D, T are defined at intermediate integer level (the "full levels"), and $\dot{\sigma}$ at interface levels (the "half levels") between these layers. Define $\dot{\sigma}_{\frac{1}{2}} = \dot{\sigma}_{NL+\frac{1}{2}} = 0$ and other $\dot{\sigma}_{k+\frac{1}{2}}$ at half levels by discretized (22) as follows

$$\dot{\sigma}_{k+\frac{1}{2}} = \sigma_{k+\frac{1}{2}} \sum_{j=1}^{NL} (D_j + \frac{\mathbf{v}_{h_j}}{p_s} \nabla p_s) \Delta \sigma_j - \sum_{j=1}^{k} (D_j + \frac{\mathbf{v}_{h_j}}{p_s} \nabla p_s) \Delta \sigma_j. \quad k = 1, 2, \cdots, NL-1. \tag{44}$$

Using discretized hydrostatic relations (20), Φ at both full and half levels can be determined as

$$\Phi_k = \frac{\Phi_{k+\frac{1}{2}} \sigma_{k+\frac{1}{2}} - \Phi_{k-\frac{1}{2}} \sigma_{k-\frac{1}{2}}}{\Delta \sigma_k} + RT_k = \Phi_{k+\frac{1}{2}} + \alpha_k RT_k, \tag{45a}$$

$$\Phi_{k+\frac{1}{2}} = \Phi_{k-\frac{1}{2}} - RT_k \ln \frac{\sigma_{k+\frac{1}{2}}}{\sigma_{k-\frac{1}{2}}} = \Phi_s + \sum_{j=k+1}^{NL} RT_j \ln \frac{\sigma_{j+\frac{1}{2}}}{\sigma_{j-\frac{1}{2}}}, \tag{45b}$$

$$\alpha_k = 1 - \frac{\sigma_{k-\frac{1}{2}}}{\Delta\sigma_k}\ln\frac{\sigma_{k+\frac{1}{2}}}{\sigma_{k-\frac{1}{2}}}, \tag{46}$$

set vertical advection item of variable X at full level as

$$(\dot{\sigma}\frac{\partial X}{\partial \sigma})_k = \frac{1}{2}\left(\dot{\sigma}_{k+\frac{1}{2}}\frac{X_{k+1}-X_k}{\Delta\sigma_{k+\frac{1}{2}}} + \dot{\sigma}_{k-\frac{1}{2}}\frac{X_k - X_{k-1}}{\Delta\sigma_{k-\frac{1}{2}}}\right),$$
$$X = U, V, T, \tag{47}$$

let vertical velocity ω at plane p satisfies

$$(\frac{\omega}{p})_k = \frac{v_{h_k}}{p_s}\cdot\nabla p_s - \alpha_k(D_k + \frac{v_{h_k}}{p_s}\cdot\nabla p_s) - \frac{\ln\sigma_{k+\frac{1}{2}}/\sigma_{k-\frac{1}{2}}}{\Delta\sigma_k}\sum_{j=1}^{k}(D_j + \frac{v_{h_j}}{p_s}\cdot\nabla p_s)\Delta\sigma_j, \tag{48}$$

then it can be verified that the satisfying of (45a, b), (47) and (48) can ensure that no false energy source or sink caused by vertical finite-difference would occur. If we further define $\sigma_{\frac{1}{2}} = 0$, then $\alpha_1 = 1$ can further ensure that no spurious angular momentum source or sink due to vertical finite-differencing arises.

Through (35a), (36) and (37), define Φ, $\dot{\sigma}\frac{\partial U}{\partial \sigma}$, $\dot{\sigma}\frac{\partial V}{\partial \sigma}$, $\dot{\sigma}\frac{\partial T}{\partial \sigma}$ and $\frac{\omega}{p}$ at full levels, we can get the grid point value of ZT, DT and TT at any full level. Since the basic variables of ζ, D, T at any level, p_s as well as geopotential height Φ_s can all be represented by finite series of spherical harmonics

$$X(\lambda, \mu, \sigma, t) = \sum_{m=-M}^{M}\sum_{l=m}^{M} X_{m,l}P_l^m(\mu)\,\mathrm{e}^{im\lambda}, \tag{49}$$

where the truncation is triangular, the maximum truncated wave number is M, the number of equally spaced longitudinal point for each latitude takes $NI = 3M + 1$ and the number of unequally spaced meridional Guassian grid point for each longitude takes $NJ = (3M+1)/2$ so as to ensure the precision of horizontal discretized computation with possible highest efficiency (Machenhauer and Rasmussen 1972; Eliasen et al. 1970). By using (49) grid-point values of ζ, D, T and p_s at n time can be obtained, by using the essential properties of spherical harmonics $(\frac{\partial X}{\partial \lambda})_m = imX_m$, $(\frac{\partial X}{\partial \mu})_m = \sum_{l=m}^{M} X_{m,l}\frac{\mathrm{d}P_l^m}{\mathrm{d}\mu}$, $X(\lambda, \mu, \sigma, t) = \sum_{m=-M}^{M} X_m(\mu, \sigma, t)\,\mathrm{e}^{im\lambda}$, $\nabla^2 P_l^m(\mu)\,\mathrm{e}^{im\lambda} = -\frac{l(l+1)}{a^2}P_l^m(\mu)\,\mathrm{e}^{im\lambda}$ we can get the grid-point values of U, V, $\frac{\partial T}{\partial \lambda}$, $\frac{\partial T}{\partial \mu}$, $\frac{\partial p_s}{\partial \lambda}$, $\frac{\partial p_s}{\partial \mu}$ and thereby Fu, Fv, ZT, DT, TT and PT at every spherical vertical level; finally, by further using

$$X_{m,l,k} = \sum_{i=1}^{NI}\sum_{j=1}^{NJ} w(\mu_j)X(\lambda_i, \mu_j, \sigma_k, t)P_l^m(\mu_j)\,\mathrm{e}^{-im\lambda}, \tag{50}$$

$ZT_{m,l,k}$, $DT_{m,l,k}$, $TT_{m,l,k}$ and $PT_{m,l,k}$ could be obtained. Here, $w(\mu)$ is Gaussian weighting factor. It is apparent that Eqs. (38) — (41) can be written both in spectral space and grid point space. Using Theorem 2 straightforwardly as described in Section II, we can get a equation

$$\Delta t^4\gamma_1(\varepsilon^n)^3 + \Delta t^2\gamma_2(\varepsilon^n)^2 + \gamma_3\varepsilon^n + \gamma_4 = 0. \tag{51}$$

Through seeking a solution ε^n of Eq. (51) with $O(\Delta t^0) = O(1)$ order of magnitudes, it can be ensured that the cubic total energy global integration conservation property (28) will retain undestroyed in the course of the semi-implicit (or explicit scheme) time differencing. This is equivalent to ensure no spurious source or sink of total energy due to the time differencing. Because only two time levels are involved, Schemes (38) — (41) will not produce the computational mode like that of traditional three time levels scheme (29) — (32), thus an artificial smoother or time filter is not needed here.

IV. COMPARATIVE EXPERIMENTS OF REAL DATA MONTHLY NUMERICAL INTEGRATION

1. *Designs of Experiments*

(1) By further performing monthly integration of the total energy conserving semi-implicit scheme using uninitialized summer FGGE data, the present work attempts to test the feasibility of long-term integration of the scheme under the complex conditions more closely approximate the actual atmosphere, and meanwhile test the capability of the scheme in preserving high-order total energy conservation in long-term integration operations. In the past, only long-time feasibility tests had been conducted respectively under idealized conditions using the relatively simplified energy and enstrophy conserving explicit scheme for a spectral model of barotropic vorticity equation and energy, enstrophy and angular momentum conserving semi-implicit scheme for a spectral model of barotropic primitive equations, although the results turned out to be satisfactory (Zhong 1992b; Zhong 1993; Zhong 1995a; Zhong 1995b).

(2) With identically idealized flat zero topography and identical real-data initial conditions, the present work also attempts to further perform comparative experiments with monthly dynamic integration between the traditional semi-implicit scheme and the total energy conserving semi-implicit fidelity scheme. The purpose is to test the contribution of the sources or sinks of the systematic errors concerning energy conservation due to the traditional time difference (called type Z systematic errors for short) to the total systematic errors and total errors, under the complex conditions of the coexisting of internal systematic errors with meteorological backgrounds more closely approximate routine operation. Meanwhile, the paper also aims to test the possibilities and potentials of improving actual medium-range even monthly forecast by formulating a total energy conserving (semi-implicit) fidelity scheme and thereby eliminating corresponding types of systematic errors. In the past, comparative experiments had been respectively conducted under idealized conditions between the relatively simplified energy conserving explicit scheme for a barotropic vorticity equation spectral model, energy conserving semi-implicit scheme for a spectral model of barotropic primitive equations and the corresponding traditional scheme, the results revealed marked differences after long-time integration (Zhong 1992b; Zhong 1995a).

Since in the above-described models there is no internal physical process involved in the atmosphere other than the pure dynamic process, no real underlying surface topography conditions, and the initial data are used without initialization, implying that the initial value of integration is far from being perfect, it is, consequently, no surprise that there exist several corresponding systematic error sources or sinks in the model of the total energy conserving fidelity scheme. If total energy conserving fidelity scheme model is set as a control standard, then it follows that, in addition to the systematic error sources in the fidelity scheme, there also exist the sources or sinks of type Z errors in the traditional model due to the failure to retain total energy conservation in the traditional semi-implicit time-difference process.

Besides, it is traditionally believed that time discrete errors are trivial as compared with spatial discrete errors and therefore are insignificant (Jiang et al. 1989). The present work, however, by retaining such noticeable errors as topography in the experiments, attempts to test whether the effects of such errors can be truly ignored when coexisting with other obvious error sources or sinks of meteorological significance.

(3) With topography included, further perform corresponding numerical integration experiments of the traditional semi-implicit scheme so as to roughly estimate the improvements obtained by removal of certain type Z errors in the traditional scheme and the improvements derived by replacing the idealized but false flat underlying surface with real topography data, and make an evaluation on the relative importance of the two. If we set the traditional model included real topography data as a control standard, then it can be said that, in addition to the systematic error sources or sinks in this model, there also exist systematic error sources or sinks (called type T errors for short) in the traditional model with idealized flat underlying surface due to the artificial substitution of real topography data with idealized flat topography.

2. *Experiment Data*

By using the new type of total energy conserving semi-implicit time-difference scheme for global spectral vertical finite-difference model of baroclinic primitive equations inclusive of idealized flat topography, the traditional semi-implicit time-difference scheme for global spectral vertical finite-difference model of baroclinic primitive equations inclusive of idealized flat topography and real model topography respectively, the present work performs three sets of numerical experiments, each having the same thirteen monthly integrations. The initial field of these integrations is FGGE data without initialization dated June 1, June 5, June 10, June 15, June 20, June 25, July 1, July 5, July 10, July 15, July 20, July 25 and August 1, respectively.

For all the calculations in this paper, $M=42$, $NL=9$, $NI=128$, $NJ=64$, $\Delta t=30$ min, $\varepsilon_f=0.05$.

3. *Results of Experiments*

(1) *Feasibility and improvements of essential properties*

Results of the 30-day numerical integration of traditional scheme and total energy conserving fidelity scheme all suggest that there are notable improvements on the essential properties of total energy and mass conservation of the new type of scheme in contrast with the traditional one (see Fig. 1). The new scheme, as experiments demonstrate, can noticeably modify the deviations from the high-order total energy and mass global integral conservation characteristics of the traditional scheme, although such deviations, increasing monotonously with the growth of integration time, are not great in quantity on monthly integral time scale. Taking the 30th day of integration for instance, these relative deviations from the conservation variables are all close to some one-ten thousandth. Besides, results of Fig. 1 undoubtedly indicate that without adopting stability methods of artificial time smoothing or filtering that have side-effects, the fidelity scheme can integrate smoothly for a long time. Theorem 2 as formulated in the paper can surely be applied to solving the formulation problems of high-order total energy conserving time-difference scheme for baroclinic primitive equations, and the high-order total energy conserving semi-implicit fidelity scheme for global spectral-vertical finite-difference model of baroclinic primitive equations based on Theorem 2 is also applicable to long-term numerical integration using real data without initialization.

(2) *The contributions of type Z and type T error to total systematic error of monthly predictions*

Errors derived from the average predictive field of a number of individual integration represent an important aspect of total errors of the model. The predictive systematic errors in the paper are referred to as errors of the 30-day average predictive field obtained from the thirteen monthly integrations.

Comparing the evolution curves of RMS error of the average field of 500 hPa geopotential height between traditional semi-implicit schemes inclusive and exclusive of real topography under identical conditions of integration, it can be noted that the contribution of type T systematic error to total systematic RMS error is, beyond doubt, very striking: it is roughly one-third of the total RMS error when total integral time reaches two weeks and reaches even two-fifths of its total amount four weeks afterwards (see Fig. 2).

Comparing the evolution curves of RMS error of 500 hPa geopotential height average field between the total energy conserving semi-implicit fidelity scheme and the traditional semi-implicit scheme exclusive of model topography under identical conditions of integration, it can be easily seen that the contribution of type Z systematic error to total RMS error can approximately reach one-third of the total RMS systematic error at the end of second week of integrations and exceeds half of its total amount four weeks afterwards (see Fig. 2).

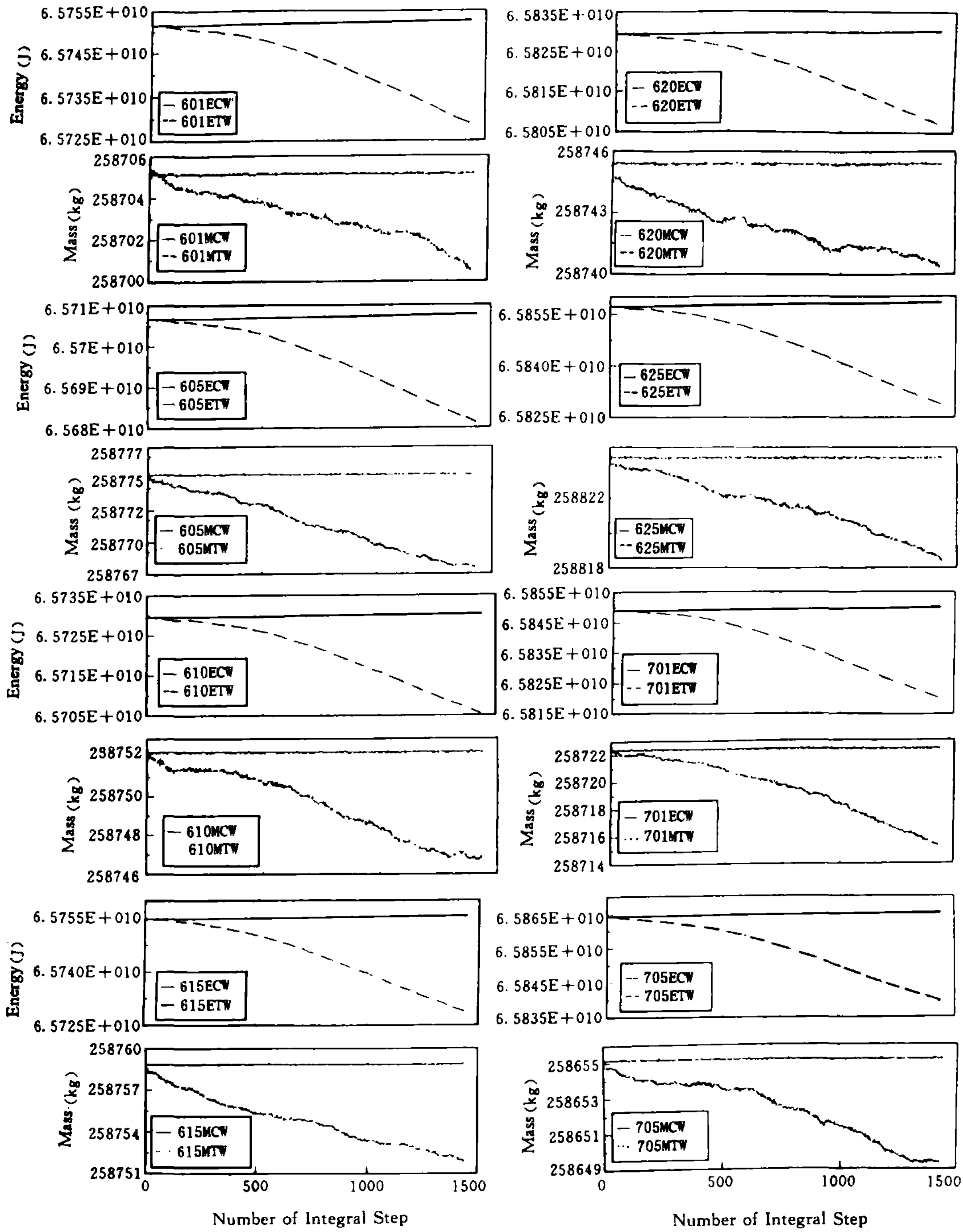
Experiments also indicate that the systematic RMS error of the fidelity scheme exclusive of topography is lower than that in the traditional scheme inclusive of topography 15 days of the integration afterwards (see Fig. 2). This implies that, as far as systematic errors are concerned, the effects of the type Z errors are, at least, in a sense, no less great than that of the type T errors.

If the type of errors in extending forecasting period can in a sense reflect the model climate systematic errors associated with climate drift, then it is by no means impossible to contribute much to the solving of climate drift by way of formulating new type of high-order total energy conserving scheme and thereby eliminating corresponding type Z systematic errors in traditional schemes.

(3) *The average improvement of total errors of monthly integration*

The averaged predictive errors derived in a number of numerical integration of a model represent the average level of the total predictive errors of the model.

With the daily average value of RMS error of 500 hPa geopotential height of the thirteen 30-day integration using the traditional semi-implicit scheme exclusive of topography (called Scheme A for short) as reference, it can be seen that, with the coexistence of multiple error sources or sinks, the average improvement of total RMS



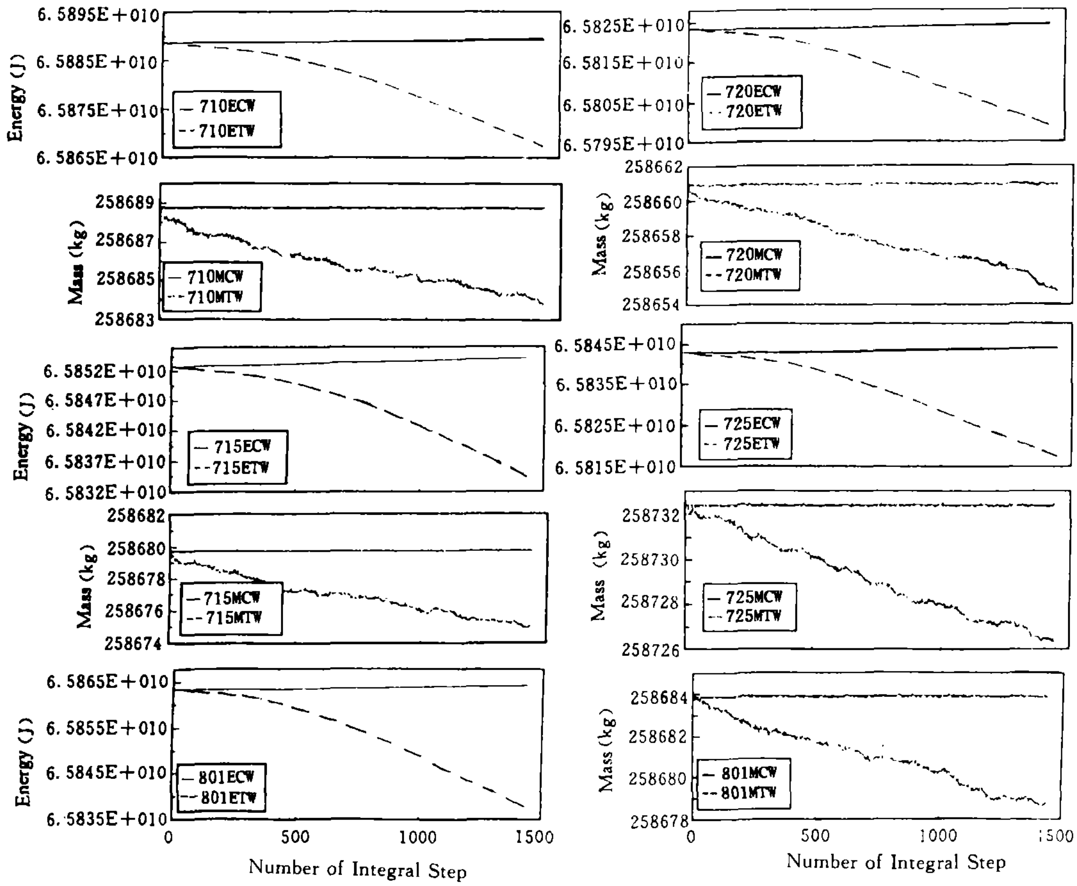


Fig. 1. The 30-day computational variation of the conservative global integration of total energy and mass of baroclinic primitive equations in the summer of 1979 (The digits in the figure represent the month and day of the initial data of each integration, for instance, 601 denotes June 1 and 715 represents July 15. Curve ECW: The computational variation of the total energy using the new fidelity scheme; Curve ETW: The computational variation of the total energy using the traditional scheme; Curve MCW: The computational variation of the mass using the new fidelity scheme; Curve MTW: The computational variation of the mass using the traditional scheme).

error obtained by formulating a total energy conserving fidelity scheme (called Scheme B for short) and thereby eliminating the corresponding type Z errors in Scheme A can reach approximately one-fourth of the total RMS errors at the end of second week of integration and even exceed over one-third of the total four weeks afterwards (see Fig. 3). This well demonstrates the great potentials of Scheme B. In fact, it can be seen that as compared with the average reduction of the total RMS errors obtained by adding model topography (called Scheme C) and thereby eliminating corresponding type T systematic errors in Scheme A, improvement with Scheme B does not surpass Scheme C until after 15 days. Prior to this point, Scheme B is obviously inferior to Scheme C (see Fig. 3). This suggests that the advantages of Scheme B primarily focus on long-term numerical integration.

In the same way, the improvements on predictions can be clearly confirmed by comparison among the three geopotential height fields with corresponding observation field deducted. Figure 4 indicates, averagely speaking, on the 30th day of thirteen integrations, for the deviations in Scheme A, Scheme B reduces over 70 percent, Scheme C no more than 45 percent in the largest negative error center (also the maximum global error center in view of absolute value) at the Antarctic Pole; in the largest positive error center at mid-latitude over the Southern Hemisphere, Scheme B reduces 25 percent, Scheme C zero percent; in the largest positive error center at mid-latitude of the Northern Hemisphere, both Scheme B and Scheme C reduce 25 percent. However, in the largest negative error center over the Arctic Pole, Scheme B reduces less than 20 percent and Scheme C more than 65 percent. Scheme C has better performance in the Northern Hemisphere while Scheme B is better in the Southern Hemisphere. This is a reasonable reflection of the notable differences between the actual underlying surface of the two hemispheres. The striking differences between the two schemes in framework and subtleties also reasonably depict the entirely different nature of the two types of errors. In general, on the 30th day of integrations, Scheme B certainly has better global performance than Scheme C. This is consistent with the calculating results of RMS errors (see Figs. 2—4).

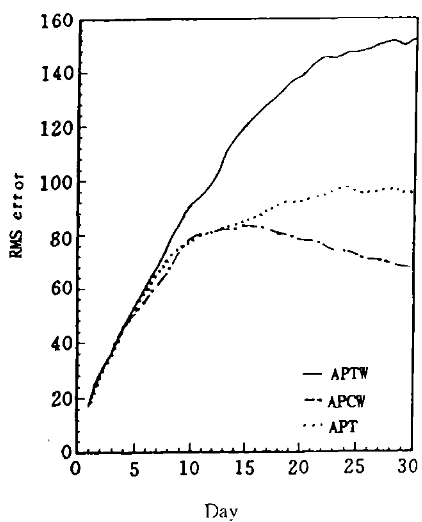


Fig. 2. The 30-day RMS error of the average of 500 hPa height of computational variation of thirteen integrations (APTW: the traditional scheme inclusive of idealized flat underlying earth's surface; APCW: the fidelity scheme inclusive of idealized flat underlying earth's surface; APT: the traditional scheme inclusive of real underlying earth's surface).

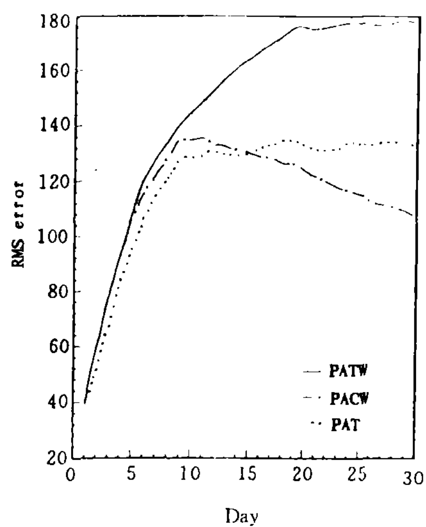
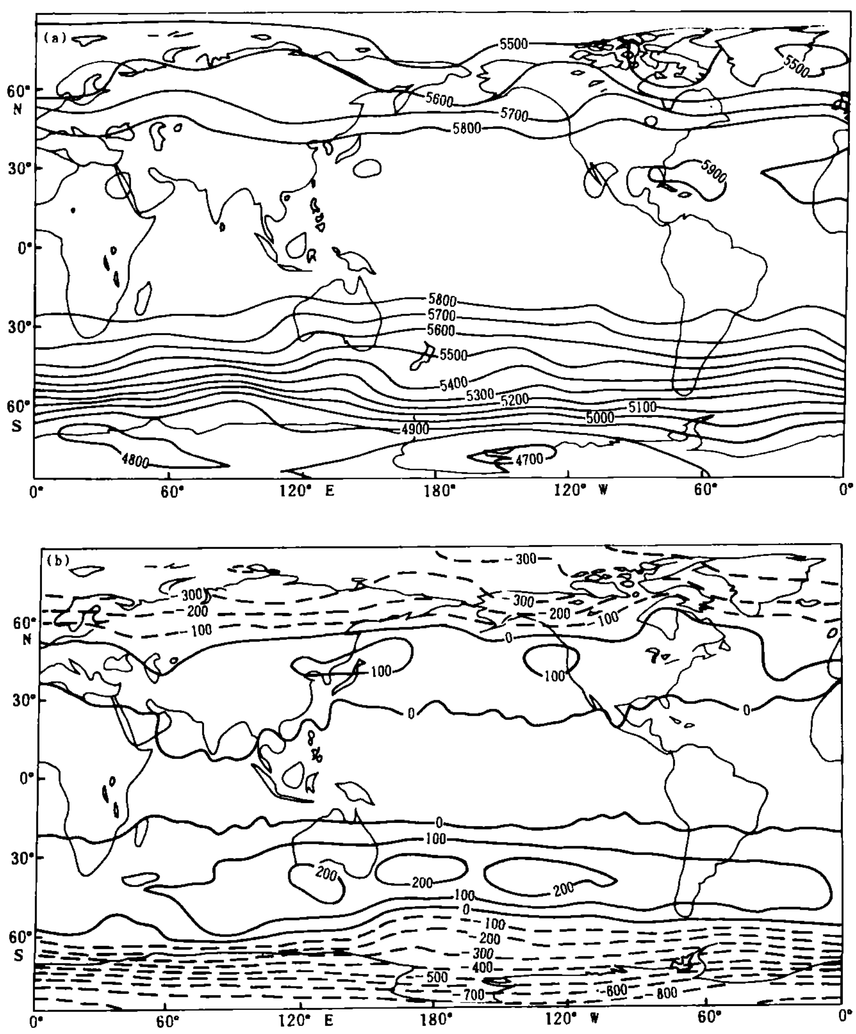


Fig. 3. The 30-day average of RMS error of 500 hPa height of computational variation of the thirteen integrations (PATW: the traditional scheme inclusive of idealized flat underlying earth's surface; PACW: the fidelity scheme inclusive of idealized flat underlying earth's surface; PAT: the traditional scheme inclusive of real underlying earth's surface).

V. SUMMARY

Physical laws of conservation are basic laws of natural world. To retain the characteristics of conservation law is a basic requirement in well-reasoned and reliable discretized computations. As has been derived in Section II of this paper, some major earlier formulation designs of conservative time-difference schemes can be directly given as special cases of the two theorems for time-difference fidelity schemes of general quadratic and cubic physical conservation law. Hence, the two mathematical theorems formulated and proved here would surely broaden the range of solvable problems and provide new mathematical basis for formulating extensive types of time-difference fidelity schemes of high-order physical conservation laws, in particular the cubic total energy conserving semi-implicit scheme for baroclinic primitive equations, thus further providing new basis and possibilities for resolving a wide range of temporal-spatial discrete fidelity schemes on the basis of previous instantly conserving spatial discrete schemes. For instance, based on



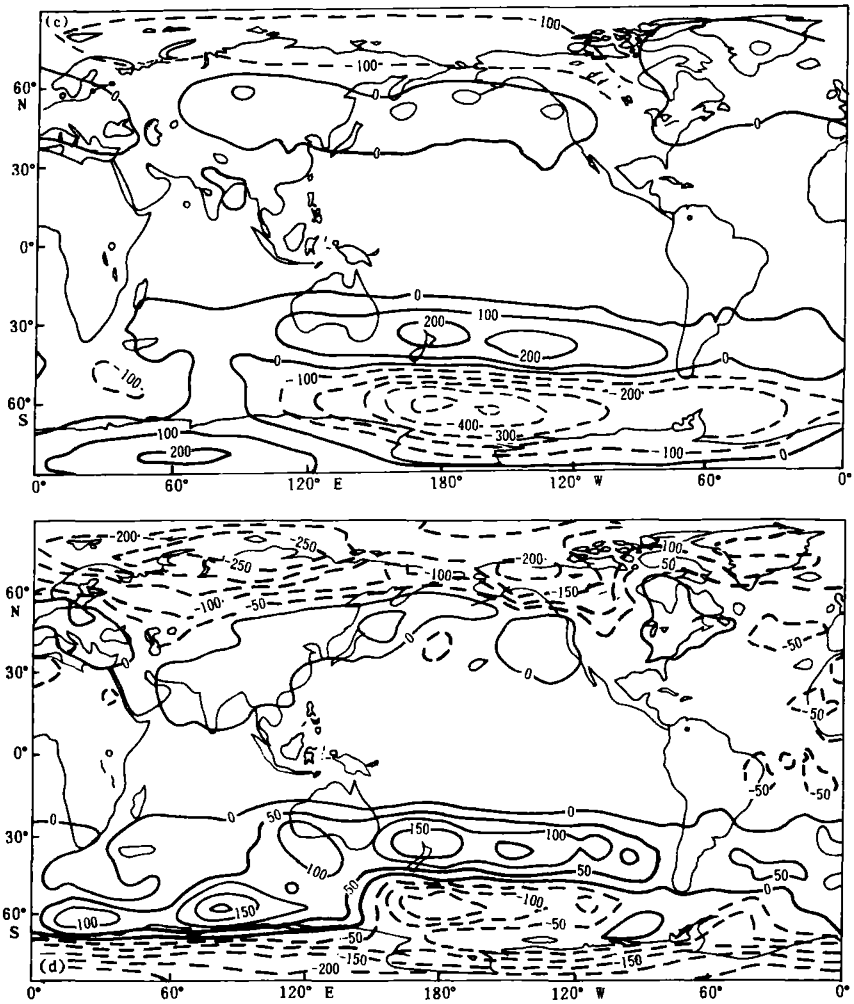


Fig. 4. The 30th day 500 hPa global average contour chart of the thirteen integration (a) Observational height; (b) Forecast height using the traditional scheme inclusive of idealized flat underlying earth's surface with observational height deducted; (c) Forecast height using the traditional scheme inclusive of real underlying earth's surface with observational height deducted; (d) Forecast height using the fidelity scheme inclusive of idealized flat underlying earth's surface with observational height deducted).

the well-known quadratic kinetic energy and enstrophy instantly conserving spatial finite-difference scheme for vorticity equation (Arakawa 1966), by using Theorem 1, a quadratic energy and a quadratic enstrophy conserving temporal-spatial finite-difference scheme for vorticity equation, never formulated before, can be worked out respectively. Also on the basis of the well-known cubic energy instantly conserving spatial finite-difference scheme for shallow water equations (Arakawa and Lamb 1981), by using Theorem 2, an energy conserving temporal-spatial finite-difference scheme for shallow water equation can be formulated. Besides, the two theorems constructed in this paper can also thoroughly solve linear and nonlinear time discrete computational instability of evolution problems with certain types of quadratic and cubic conserving integral properties, and by combining earlier these types of quadratic or cubic instantly conserving spatial discrete schemes, thoroughly solve their linear and nonlinear temporal-spatial discrete computational instability.

The traditional global spectral-vertical finite-difference semi-implicit model for baroclinic primitive equations is being employed by many countries in the world for operational forecast and general circulation simulation. For various reasons, the basic formulation problems of total energy conserving semi-implicit scheme for baroclinic primitive equations remain unsolved for long. The present work demonstrates that Theorem 2 as formulated in the paper can surely be applied to solving this problem, and the high-order total energy conserving semi-implicit fidelity scheme for global spectral-vertical finite-difference model of baroclinic primitive equations formulated by Theorem 2 is also applicable to real data long-term numerical integrations.

Thirteen FGGE data month-integration numerical experiments indicate that, the new type total energy conserving semi-implicit fidelity scheme can certainly modify the systematic deviations of energy and mass conservation characteristics of the traditional scheme. It should be particularly noted that, under the same experimental conditions of the present work, the type Z systematic error source or sink created by the violation of physical conservation law in the time difference process of the traditional model can contribute up to one-third of the total RMS systematic error at the end of the second week of integration and exceeds one half of the total amount four weeks afterwards. In contrast, by realizing a total energy conserving semi-implicit fidelity scheme and thereby eliminating the corresponding type Z errors, roughly an average of one-fourth of the RMS error in the traditional forecasting can be reduced at the end of the second week of integration, and averagely more than one-third reduced four weeks afterwards. The potentials of the total energy conserving fidelity scheme can be great for improving traditional medium-range and even monthly weather predictions.

Experiments in the present work also show that type Z errors due to traditional time-difference are far from notable in quantity (see Fig. 1) and it seems less significant to eliminate the negative effects of these small deviations. Results of the experiments, however, prove just the opposite. In fact, if we look at it from another angle, these results are by no means strange. Evidently, the actual atmospheric system is a complex nonlinear system with sensitivity of initial value. Since type Z errors are internal errors available at all integral time, its total accumulative effects, without doubt, can be very great.

The formulation of (high-order) conserving scheme is usually believed to be of great or even remarkable importance in problems requiring long-term numerical integration. However, up till now, not enough experiments, particularly those of conserving time-difference schemes, seem to support this statement. Hence, it is, strictly speaking, merely a theoretical hypothesis. Moreover, there is no conclusion whatsoever as to how long it will take before this type of computational design becomes significant. Experiments of the present work indicate that, if we assume the reduction of total systematic RMS error by half and total RMS error by one-third as significant, then this hypothesis would undoubtedly hold true; if it can be said as significant to reduce total systematic RMS error by one-third or less and total RMS error by one-fourth or less, then the "long term" could mean month, two weeks or still shorter.

In a sense, the negative effects of type Z interior systematic errors due to traditional time-difference on the total systematic errors are no less great than that of type T external systematic errors due to the artificial substituting of real topography data with idealized zero topography. Such a result would indeed be very helpful for us to better understand the significance of constructing high-order conserving time-difference schemes.

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