STUDY OF NONLINEAR ROSSBY WAVES IN THE ATMOSPHERE UNDER SEMI-GEOSTROPHIC APPROXIMATION *

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ABSTRACT

Under semi-geostrophic approximation the nonlinear ordinary differential equations are obtained for the motion in the barotropic and baroclinic atmospheres with the effects of zonal shear basic flow and topographic forcing included. Two constraints are acquired of finite-amplitude periodic and solitary waves in the original model with the aid of the phase-plane geometric qualitative theory of a dynamic system defined by the differential equation. The explicit solution of the nonlinear waves is found by means of the approximation method and some significant results are achieved.

Key words: semi-geostrophic, dynamic system, phase plane, nonlinear Rossby waves

I. INTRODUCTION

In recent years much progress has been made in the research of nonlinear waves. The selection of small parameters is, however, arbitrary and done in a complicated manner when using the multi-scale perturbation method. To seek for a simpler technique, part of nonlinear property is retained in the vorticity equation and then the behavior of nonlinear Rossby waves around the mechanically-equilibrium point is examined (Liu et al., 1982a; b). Wu (1985) indicated the application of longwave approximation in the study of Rossby waves. Liu et al. (1987) reported that the nonlinear ordinary differential equation is changed to the KdV form in terms of a complete nonlinear vorticity equation and the principle of semi-geostrophic approximation, indicating that the Rossby waves under the condition possess all nonlinear characteristics, thus providing a useful approach to the problem of nonlinear waves.

A qualitative analysis is performed of nonlinear motion by virtue of the features of phase space in relation to the nonlinear dynamic system defined by the ordinary differential equation. The presence of the waves is shown by the existence of a periodic solution of the nonlinear equation and conditions of the presence of the periodic and solitary forms of the Rossby waves are obtained by those of the existence of a closed phase orbit over the plane in association with the dynamic system. Investigated, separately, are the conditions of the presence of periodic and solitary waves, and the nature of the solutions in the barotropic and baroclinic atmospheres with the effects of zonal flow and topographic forcing included, leading to some results of significance.

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II. DYNAMIC SYSTEM AND MOTION INVARIANTS IN THE BAROTROPIC AND BAROCLINIC ATMOSPHERES

1. *The Barotropic Model*

The shallow–water, or barotropic, model for describing atmospheric motions is in the form

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+u\dfrac{\partial}{\partial x}+v\dfrac{\partial}{\partial y}\right)u-fv=-g\dfrac{\partial h}{\partial x}, \\ \left(\dfrac{\partial}{\partial t}+u\dfrac{\partial}{\partial x}+v\dfrac{\partial}{\partial y}\right)v+fu=-g\dfrac{\partial h}{\partial y}, \\ \left(\dfrac{\partial}{\partial t}+u\dfrac{\partial}{\partial x}+v\dfrac{\partial}{\partial y}\right)(h-h_B)+(h-h_B)\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)=0, \end{cases} \tag{1}$$

where h and h_B are the height of the free surface and terrain, respectively, with $h_B \ll h$.

Set

$$u=\bar{u}(y)+u', \quad v=v', \quad h=\bar{h}(y)+h', \tag{2}$$

where $\bar{u}(y)=-(g/f_0)(\partial\bar{h}/\partial y)$.

Substituting Eq.(2) into Eq.(1), and then dropping the superscript "prime" and using the condition of $h_B \ll h$, we find

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\bar{u}\dfrac{\partial}{\partial x}+u\dfrac{\partial}{\partial x}+v\dfrac{\partial}{\partial y}\right)u-\left(f-\dfrac{\partial\bar{u}}{\partial y}\right)v=-g\dfrac{\partial h}{\partial x}, \\ \left(\dfrac{\partial}{\partial t}+\bar{u}\dfrac{\partial}{\partial x}+u\dfrac{\partial}{\partial x}+v\dfrac{\partial}{\partial y}\right)v+fu=-g\dfrac{\partial h}{\partial y}, \\ \left(\dfrac{\partial}{\partial t}+\bar{u}\dfrac{\partial}{\partial x}+u\dfrac{\partial}{\partial x}+v\dfrac{\partial}{\partial y}\right)h+v\dfrac{\partial\bar{h}}{\partial y}+(\bar{h}+h)\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)-\left(u\dfrac{\partial h_B}{\partial x}+v\dfrac{\partial h_B}{\partial y}\right)=0. \end{cases} \tag{3}$$

Changing the equation of motion of (3) into a vorticity one, we have

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(f_0-\frac{\partial\bar{u}}{\partial y}+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$
$$+\left(\beta-\frac{\partial^2\bar{u}}{\partial y^2}\right)v=0. \tag{4}$$

Under semi–geostrophic approximation, Eq.(4) and the equation of continuity of (3) can be rewritten as

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\bar{u}\dfrac{\partial}{\partial x}+u\dfrac{\partial}{\partial x}+v\dfrac{\partial}{\partial y}\right)\xi_g+(\bar{\xi}_a+\xi_g)\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)+\left(\beta-\dfrac{\partial^2\bar{u}}{\partial y^2}\right)v_g=0, \\ \left(\dfrac{\partial}{\partial t}+u\dfrac{\partial}{\partial x}+v\dfrac{\partial}{\partial y}\right)h+(\bar{h}+h)\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)-\left(v_g\dfrac{\partial h_B}{\partial y}+u_g\dfrac{\partial h_B}{\partial x}\right)=0, \end{cases} \tag{5}$$

where ξ_g $(=\partial v_g/\partial x-\partial u_g/\partial y)$ denotes the geostrophic vorticity, $\bar{\xi}_a=f_0-\partial\bar{u}/\partial y$, $u_g=(-g/f_0)(\partial h/\partial y)$ and $v_g=(g/f_0)(\partial h/\partial x)$. It is noted that u_g and v_g are used in place of u and v in $(\beta-\partial^2\bar{u}/\partial y^2)v$, $(\partial h_B/\partial y)v$ and $(\partial h_B/\partial x)u$ (Liu et al., 1988). Eq.(5) describes Rossby waves only. Assume it to have the formal solution

$$u=U(\theta), \quad v=V(\theta), \quad gh=\Phi(\theta), \qquad (\theta=kx+ly-rt) \tag{6}$$

where θ represents the phase angle, k and l the wavenumber in the x and y directions, respectively, and r the circular frequency. By inserting Eq.(6) into Eq.(5) and setting the terrain slope to be constant and the domain of y variation not to be large so as to have the parameters vary over a smaller range, as shown in Zhao et al. (1991), we obtain the equation satisfying Φ of the form

$$\Phi'' = \frac{\left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2}\right)\Phi^2 + \left(f_0\bar{\xi}_a C_x + f_0\bar{\xi}_a C_h + C_0^2\beta - C_0^2\frac{\partial^2 \bar{u}}{\partial y^2}\right)\Phi}{K_h^2 C_0^2 (C_x - \bar{u}) - K_h^2(\bar{u} + C_h)\Phi}, \tag{7}$$

where $C_0^2 = g\bar{h}$, wave velocity $C_x = r/k$, $K_h^2 = k^2 + l^2$ and $C_h = (g/f_0)(\partial h_B/\partial y) - (lg/f_0 k)(\partial h_B/\partial x)$. The primed quantity denotes the differentiation with respect to the phase angle.

Eq.(7) represents the second-order nonlinear ordinary differential equation describing nonlinear barotropic Rossby waves under the effects of zonal basic flow and terrain. For $\bar{u} + C_h = 0$ no effects of the factors are available, in which case Eq.(7) is rewritten as

$$\Phi'' = \frac{\left(\beta - \frac{\partial^2 \bar{u}}{\partial y^2}\right)\Phi^2 + \left(f_0\bar{\xi}_a C_x + f_0\bar{\xi}_a C_h + C_0^2\beta - C_0^2\frac{\partial^2 \bar{u}}{\partial y^2}\right)\Phi}{K_h^2 C_0^2 (C_h + C_x)}, \tag{8}$$

which is easy to transform into a KdV equation with the related solutions of cnoidal and solitary waves. Studies of the nature of KdV wave solutions are numerous and can be found elsewhere. This article has focus on the case of $\bar{u} + C_h \neq 0$.

For $\beta - \frac{\partial^2 \bar{u}}{\partial y^2} \neq 0$, Eq.(7) reads

$$\frac{d^2\Phi}{d\tau^2} = \mathrm{sgn}(e)\frac{\Phi^2 + A\Phi}{\Phi - B}, \tag{9}$$

where

$$\begin{cases} A = \dfrac{f_0\bar{\xi}_a(C_x + C_h)}{\beta - \partial^2\bar{u}/\partial y^2} + C_0^2, & B = \dfrac{C_0^2(C_x - \bar{u})}{\bar{u} + C_h}, \\ e = -\dfrac{\beta - \partial^2\bar{u}/\partial y^2}{K_h^2(\bar{u} + C_h)}, & \tau = |e|^{\frac{1}{2}}\theta, \\ \mathrm{sgn}(e) = \begin{cases} 1, & \text{for } e > 0 \\ -1, & \text{for } e < 0. \end{cases} \end{cases} \tag{10}$$

Eq.(9) can be changed to the form

$$\frac{d^2 Q}{d\tau^2} = \mathrm{sgn}(e)\left(Q + \frac{a}{b^2}\frac{1}{Q - 1}\right), \tag{11}$$

where $Q = (\Phi + A + B)/(A + 2B)$, $a = (A + B)B$, $b = A + 2B$.

Eq.(11) is equivalent to the following differential dynamic system

$$\begin{cases} \dot{Q} = P = \dfrac{\partial H}{\partial P}, \\ \dot{P} = \operatorname{sgn}(e)\left(Q + \dfrac{a}{b^2}\dfrac{1}{Q-1}\right) = -\dfrac{\partial H}{\partial Q}, \end{cases} \tag{12}$$

where the superscript "dot" indicates the differentiation with respect to τ, and

$$H = \frac{P^2}{2} - \operatorname{sgn}(e)\left(\frac{Q^2}{2} + \frac{a}{b^2}\ln|Q-1|\right). \tag{13}$$

Evidently, Eq.(12) is the equation of Hamilton type such that the dynamic system relative to the original equation is conservative with no attractors of any type available except the likelihood of unchanged flow pattern, i.e., the motion invariant. Then we have

$$\dot{H} = \frac{\partial H}{\partial P}\dot{P} + \frac{\partial H}{\partial Q}\dot{Q} = 0, \tag{14}$$

so that H, being independent of τ, denotes a motion invariant. Then we find

$$\frac{\dot{Q}^2}{2} - \operatorname{sgn}(e)\left(\frac{Q^2}{2} + \frac{a}{b^2}\ln|Q-1|\right) = \frac{P^2}{2} + V(Q) = H_0(\tau = 0), \tag{15}$$

where $V(Q)$ is equivalent to potential energy and H_0 is specified by the initial disturbance. Thus, the problem at issue is reduced to that of the effect upon H_0 associated with initial disturbance of the unit mass in the potential field defined by $V(Q)$.

Now we come to investigate the case of $\beta - \partial^2\bar{u}/\partial y^2 = 0$, wherein Eq.(7) is rewritten as

$$\frac{d^2Q}{d\tau^2} = \operatorname{sgn}(e_1)\frac{Q}{Q+1}, \tag{16}$$

where

$$\begin{cases} Q = \dfrac{\Phi(\bar{u} + C_h)}{C_0^2(\bar{u} - C_x)}, \\ e_1 = \dfrac{f_0\bar{\xi}_a(C_x + C_h)}{K_h^2 C_0^2(C_x - \bar{u})}, \\ \tau = |e_1|^{1/2}\theta. \end{cases} \tag{17}$$

Eq.(16) is equivalent to the following differential dynamic system

$$\begin{cases} \dot{Q} = P = \dfrac{\partial H}{\partial P}, \\ \dot{P} = \operatorname{sgn}(e_1)\dfrac{Q}{Q+1} = -\dfrac{\partial H}{\partial Q}, \end{cases} \tag{18}$$

in which $H = P^2/2 - \operatorname{sgn}(e_1)(Q - \ln|Q+1|)$. Similarly, we have the motion invariant in the form

$$\frac{\dot{Q}^2}{2} - \operatorname{sgn}(e_1)(Q - \ln|Q+1|) = \frac{P^2}{2} + V(Q) = H_0(\tau = 0). \tag{19}$$

2. *The Baroclinic Model*

Under semi-geostrophic approximation the equations governing the motion in the baroclinic atmosphere take the form

$$\begin{cases} (\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})\xi_g+\beta v_g-\frac{f_0}{\rho_0}\frac{\partial(\rho_0 w)}{\partial z}=0, \\ (\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})\frac{\partial\varphi}{\partial z}+N^2 w=0, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{\rho_0}\frac{\partial(\rho_0 w)}{\partial z}=0, \\ f_0(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y})=\nabla^2\varphi, \end{cases} \tag{20}$$

in which the meanings of these symbols are the same as in Liu et al. (1987). Assume the solution of traveling waves to be in the form

$$\begin{cases} u=U(\theta), & v=V(\theta) \\ w=W(\theta), & \varphi=\Phi(\theta) \end{cases} \quad \text{with } \theta=kx+ly+nz-rt. \tag{21}$$

Substitution of (21) into (20) with $\beta k / r(K_h^2+n_1^2)$ in lieu of Φ'' as shown in Liu et al. (1987) yields the equation satisfying Φ in the form

$$-r\left[K_h^2\frac{1-(\frac{n}{N})^2\beta\Phi}{C_x(K_h^2+n_1^2)}+n_1^2\right]\Phi'''+\beta k\left[\frac{1-(\frac{n}{N})^2\beta\Phi}{C_x(K_h^2+n_1^2)}\right]^2\Phi'=0. \tag{22}$$

Integration done of Eq.(22) with respect to θ with the integration constant of zero gives

$$\Phi''-\frac{b}{s}\left[\frac{(1-s\Phi)^2}{2}-d^2(1-s\Phi)+d^4\ln|1-s\Phi+d^2|\right]=0, \tag{23}$$

where $s=(n/N)^2\beta/C_x(K_h^2+n_1^2)$, $b=-\beta/C_xK_h^2$, $d^2=(n_1/K_h)^2$ and $n_1^2=f_0^2(n/N)^2$.

With $Q=1-s\Phi$ and $\tau=|b|^{1/2}\theta$, Eq.(23) is changed to

$$\frac{d^2Q}{d\tau^2}+\operatorname{sgn}(b)(\frac{Q^2}{2}-d^2Q+d^4\ln|Q+d^2|)=0, \tag{24}$$

which is equivalent to the following differential dynamic system

$$\begin{cases} \dot{Q}=P=\frac{\partial H}{\partial P}, \\ \dot{P}=-\operatorname{sgn}(b)(\frac{Q^2}{2}-d^2Q+d^4\ln|Q+d^2|)=-\frac{\partial H}{\partial Q}, \end{cases} \tag{25}$$

where $H=P^2/2+\operatorname{sgn}(b)[Q^3/6-d^2Q^2/2-d^4Q+d^4(Q+d^2)\ln|Q+d^2|]$. Similarly, we have the motion invariant in the form

$$\frac{\dot{Q}^2}{2}+\operatorname{sgn}(b)\left[\frac{Q^3}{6}-\frac{d^2Q^2}{2}-d^4Q+d^4(Q+d^2)\ln|Q+d^2|\right]=\frac{P^2}{2}+V(Q)=H_0(\tau=0). \tag{26}$$

Up to this point we have obtained the dynamic systems in relation to the barotropic and baroclinic models together with the motion invariants.

III. ANALYSIS OF CHARACTERISTICS OF THE MOTION ON THE PHASE PLANE ASSOCIATED WITH THE DYNAMIC SYSTEM

The characteristics of the motion over the phase plane in relation to the dynamic system are studied in terms of the motion invariants. This proves to be a useful approach. Obviously, the closed (homoclinic) trajectory on the plane corresponds to the solution of finite-amplitude periodic (solitary) waves. In the barotropic model, for $e \neq 0$ two equilibrium points A_2 $((A+B)/(A+2B),\ 0)$ and A_1 $(B/(A+2B),\ 0)$ exist on the $(Q,\ P)$-defined phase plane (Fig. 1) and only one point $(0,\ 0)$ for $e=0$ (Fig. 2). And in the baroclinic model two points A_2 $(a_2,\ 0)$ and A_1 $(a_1,\ 0)$ are seen (Fig. 3), where a_1 and a_2 are two single real roots (with $a_2 > -d^2 > a_1$) of the transcendental equation

$$\frac{Q^2}{2} - d^2 Q + d^4 \ln\left|Q + d^2\right| = 0. \tag{27}$$

From the pattern of the phase trajectory (PT) we have the following.

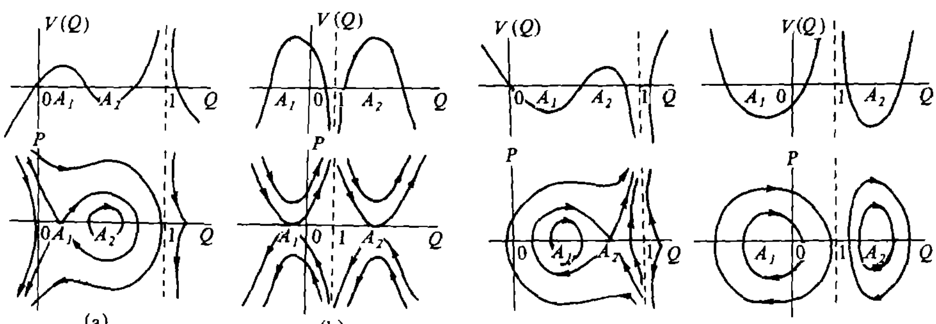

Fig. 1. Potential function $V(Q)$ and phase trajectory for $e \neq 0$ in the barotropic model with (a) $\mathrm{sgn}(e)=1$ and $a>0$. (b) $\mathrm{sgn}(e)=1$ and $a<0$; (c) $\mathrm{sgn}(e)=-1$ and $a>0$ and (d) $\mathrm{sgn}(e)=-1$ and $a<0$.

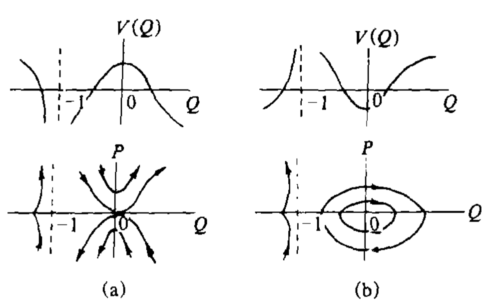

Fig. 2. As in Fig. 1 but for $e=0$ with (a) $\mathrm{sgn}(e_1)=1$, and (b) $\mathrm{sgn}(e_1)=-1$ only.

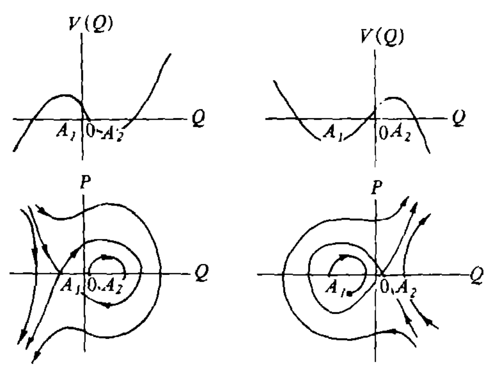

Fig. 3. As in Fig. 1 except for the baroclinic model with (a) $\mathrm{sgn}(b)=1$ and (b) $\mathrm{sgn}(b)=-1$ only.

(1) With $\mathrm{sgn}(e)=1$ and $a>0$ (Fig. 1a) and with the initial disturbance H_0 satisfying $H_1< H_0< H_2$ where $H_1=\frac{1}{2}(1-\frac{A}{b})-\frac{a}{b^2}(1-2\ln\frac{1+A/b}{2})$ and $H_2=\frac{1}{2}(1+\frac{A}{b})-\frac{a}{b^2}(1-2\ln\frac{1-A/b}{2})$, the PT is a closed curve around the center A_2 or a homoclinic trajectory through the saddle point A_1. And for $H_0> H_2$ the PT is unclosed.

(2) With $\mathrm{sgn}(e)=1$ and $a<0$ (Fig. 1b) no closed trajectories are existent, meaning neither finite–amplitude periodic (FAP) nor solitary waves, for any initial disturbance H_0.

(3) With $\mathrm{sgn}(e)=-1$ and $a>0$ (Fig. 1c) the PT is a closed curve around the center A_1 or a homoclinic trajectory through the saddle point A_2 when $-H_2<H_0<-H_1$ is satisfied, and the PT is unclosed, suggesting that neither FAP nor solitary waves are available, for $H_0>-H_1$.

(4) With $\mathrm{sgn}(e)=-1$ and $a<0$ (Fig. 1d), for any H_0 the PT's are two families of closed curves about A_1 and A_2, separately, corresponding to the solution of FAP waves but no solution of solitary waves is found.

(5) With $\mathrm{sgn}(e_1)=1$ (Fig. 2a) no closed trajectory exists, indicating no solution of either FAP or solitary waves, for any H_0.

(6) With $\mathrm{sgn}(e_1)=-1$ (Fig. 2b) and for $H_0\geqslant 0$, the PT is a closed curve about the center which is related to the solution of FAP waves but no solitary waves are available, and for $H_0<0$ the PT is unclosed with neither of the types existent.

(7) With $\mathrm{sgn}(b)=1$ (Fig. 3a) and for $a_1^3/3<H_0\leqslant a_2^3/3$ the PT is a closed curve around A_2 or a homoclinic trajectory through A_1 and for $H_0>a_2^3/3$ the PT is unclosed so that no solution of bounded steady waves is existent.

(8) With $\mathrm{sgn}(b)=-1$ (Fig. 3b) the PT is a closed curve surrounding A_1 or a homoclinic trajectory through A_2 for $-a_2^3/3<H_0\leqslant -a_1^3/3$, and the PT is unclosed so that no solution of bounded steady waves is existent for $H_0>-a_1^3/3$.

IV. CONDITIONS OF THE EXISTENCE OF SOLUTIONS OF NONLINEAR WAVES

The westerly basic flow $\bar{u}>0$ is considered in the barotropic model. From Section III we can have the conditions of the presence of the following nonlinear waves and their types.

(1) If $(\beta-\frac{\partial^2\bar{u}}{\partial y^2})/(\bar{u}+C_h)<0$, then

$$\left(\frac{f_0\xi_a}{\beta-\frac{\partial^2\bar{u}}{\partial y^2}}+\frac{C_0^2}{\bar{u}+C_h}\right)\frac{(C_x+C_h)(C_x-\bar{u})}{\bar{u}+C_h}>0. \tag{28}$$

For the initial phase disturbance $H_1<H_0\leqslant H_2$ there is the solution of FAP or solitary waves (for $H_0=H_2$). and if $H_0>H_2$ or

$$\left(\frac{f_0\xi_a}{\beta-\frac{\partial^2\bar{u}}{\partial y^2}}+\frac{C_0^2}{\bar{u}+C_h}\right)\frac{(C_x+C_h)(C_x-\bar{u})}{\bar{u}+C_h}<0 \tag{29}$$

is satisfied, then no solution of nonlinear Rossby waves is available for any H_0.

(2) For $(\beta - \frac{\partial^2 \bar{u}}{\partial y^2})/(\bar{u} + C_h) > 0$, if Eq.(28) holds, then, for $-H_2 < H_0 \leqslant -H_1$, we find the solution of FAP or solitary waves (with $H_0 = -H_1$). And for $H_0 > -H_1$ no solution of nonlinear Rossby waves is existent. If Eq.(29) holds, then the solution of FAP waves is available but no solution of the other type is existent for any H_0.

(3) If $\beta - \frac{\partial^2 \bar{u}}{\partial y^2} = 0$, then

$$\bar{\xi}_a (C_x + C_h)(C_x - \bar{u}) < 0, \tag{30}$$

which has the solution of FAP waves but none of the other types when $H_0 \geqslant 0$. If $H_0 < 0$ or

$$\bar{\xi}_a (C_x + C_h)(C_x - \bar{u}) > 0 \tag{31}$$

is satisfied, then, for any H_0 there is no solution of nonlinear Rossby waves.

From the foregoing discussion we see that in the original model there exist two basic constraints, i.e., initial phase disturbance (H_0) and phase velocity (C_x), for the solutions of FAP and solitary waves. The effects of basic flow, its shear and terrain on the constraints are displayed in determining the possible domain of evaluating H_0 and C_x. Therefore, the flow, shear and terrain have some influence upon wave behavior. To stress the topographic effect on the waves the zonal flow and β effects are neglected, i.e., $\beta - \partial^2 \bar{u}/\partial y^2 = 0$. For $H_0 \geqslant 0$, only when $C_x(C_x+C_h) < 0$ is the solution of topographic FAP waves available. As such, for $C_h > 0$ ($C_h < 0$) we have $-C_h < C_x < 0$ ($0 < C_x < C_h$). It is seen that in the former case the west (east)-facing slope favoring the trailing (leading) waves, the orographic waves generated over the southern slope propagate westward, and in the latter, western (eastern) slope promoting leading (trailing) waves, the orographic waves produced on the north-facing slope move eastward. As a rule, Rossby waves march towards the east if westerly flow is taken into account, the northern (southern) slope causing the speeding up (slowing down) of the propagation (a result in agreement with the observational fact) so that the E-W directed terrain favors the formation of a transversal trough or a shear line. On the other hand, the western (eastern) slope quickens leading (trailing) waves and slows down trailing (leading) ones so that N-S oriented terrain promotes the shortening (elongating) of the leading (trailing) waves. Now we consider the effect of the factor f. The N-S directed terrain makes it possible for the leading wave pattern to change to a closed circulation of low pressure, which seems to serve as an interpretation of the vortex forming on the east side of the Qinghai-Xizang Plateau, as is given by a quasi-geostrophic model (Lu, 1986). In addition, from (30) of the present work one can see that the result achieved is true only in the inertially stable domain ($\bar{\xi}_a > 0$) and for ($\bar{\xi}_a < 0$) it is somewhat different. For $C_h > 0$ ($C_h < 0$) we have $C_x > 0$ ($C_x < 0$) or $C_x < -C_h$ ($C_x > -C_h$). Therefore, the trailing (leading) waves on the west (east)-facing slope can travel either westward or eastward along the southern slope, and so can the leading (trailing) waves on the western (eastern) slope along the north-facing slope. In a word, the terrain has effect on the waves in a more complicated fashion with basic flow and shear considered than without. It is seen from the conditions of nonlinear waves that effects of zonal-shear flow and terrain determine the existence of the solutions and forms (periodic or solitary) and the domain of H_0 evaluation.

Now we consider the baroclinic model. With $\mathrm{sgn}(b) = \pm 1$ and H_0 satisfying certain conditions there is likely to be FAP or solitary waves. From the foregoing analysis we have the following discriminant conditions.

(4) For $b>0$, i.e., $C_x<0$, with $a_1^3/3<H_0\leqslant a_2^3/3$ there exist FAP or solitary waves (with $H_0=a_2^3/3$). And for $H_0>a_2^3/3$ no nonlinear bounded Rossby waves are available.

(5) For $b<0$, or $C_x>0$, with $-a_2^3/3<H_0\leqslant -a_1^3/3$ there are FAP or solitary waves (with $H_0=-a_1^3/3$) as well. And for $H_0>-a_1^3/3$ no nonlinear bounded Rossby waves exist.

Clearly, Rossby waves appearing at $C_x>0$ are not existent under linear or quasi-geostrophic approximation but the product of nonlinear effects only. Hence, semi-geostrophic approximation is an approach to the study of nonlinear problem, for it is able to show the nonlinear features of the system to full advantage.

V. EXPLICIT APPROXIMATE SOLUTION OF NONLINEAR ROSSBY WAVES

With the conditions satisfied for nonlinear waves, the implicit expression for the wave solution can be found by means of the motion invariant, viz.,

$$\tau=\pm\int[H_0-V(Q)]^{-\frac{1}{2}}dQ. \tag{32}$$

To seek for the explicit approximate solution, the function approximation technique (Huang et al., 1987) is employed for the barotropic model with $\operatorname{sgn}(e)=1$ and $a>0$ as the example. It is evident for $H_1<H_0<H_2$ the related PT is a closed curve around A_2, in which Q has a bounded periodic solution for τ. A cubic polynomial function is used to make approximation to $V(Q)=H_0$ with the polynomial having the same number and value ($Q<1$) of zero points as $V(Q)=H_0$. Thus, the solution of steady bounded waves of Φ is found by integrating with the aid of Eq.(32), namely,

$$\Phi=\left[\frac{f_0\bar{\xi}_a(C_x+C_h)}{\beta-\frac{\partial^2\bar{u}}{\partial y^2}}+C_0^2+\frac{2C_0^2(C_x-\bar{u})}{\bar{u}+C_h}\right]\left[Q_2+(Q_1-Q_2)Cn^2\left(\frac{Q_1-Q_3}{6}\frac{\frac{\partial^2\bar{u}}{\partial y^2}-\beta}{\bar{u}+C_h}\right)^{\frac{1}{2}}\frac{\theta}{K_h}\right]$$
$$-\left(\frac{f_0\bar{\xi}_a}{\beta-\frac{\partial^2\bar{u}}{\partial y^2}}+\frac{C_0^2}{\bar{u}+C_h}\right)(C_x+C_h), \tag{33}$$

where Q_i ($i=1,2,3$) are three single roots of value <1 with $Q_1>Q_2>Q_3$ of the transcendental equation

$$-\left(\frac{Q^2}{2}+\frac{a}{b^2}\ln|Q-1|\right)=H_0. \tag{34}$$

Evidently (33) gives cnoidal waves. For $H_0=H_2$ the PT is a homoclinic trajectory through the saddle point A_1. For Q satisfying $Q_2<Q$ ($\tau=0$)$\leqslant Q_1$ in relation to the initial phase, we find the solution of solitary waves:

$$\Phi = \left[\frac{f_0\bar{\xi}_a(C_x + C_h)}{\beta - \frac{\partial^2 \bar{u}}{\partial y^2}} + C_0^2 + \frac{2C_0^2(C_x - \bar{u})}{\bar{u} + C_h}\right]\left[Q_2 + (Q_1 - Q_2)\sec^2 h\left(\frac{Q_1 - Q_3}{6}\frac{\frac{\partial^2 \bar{u}}{\partial y^2} - \beta}{\bar{u} + C_h}\right)^{\frac{1}{2}}\frac{\theta}{K_h}\right]$$

$$- \left(\frac{f_0\bar{\xi}_a}{\beta - \frac{\partial^2 \bar{u}}{\partial y^2}} + \frac{C_0^2}{\bar{u} + C_h}\right)(C_x + C_h), \tag{35}$$

where Q_1 is the single root, and $Q_2 = Q_3$, the repeated roots of Eq.(34).

To investigate the solutions in the baroclinic model we take the case of sgn(b) = 1 for example. With $a_1^3 / 3 < H_0 < a_2^3 / 3$, Q has a periodic solution for τ. In the way as mentioned before we achieve the solution of steady bounded waves of Φ in the form

$$\Phi = \frac{C_x}{\beta}(K_h^2 + n_1^2)\left(\frac{N}{n}\right)^2\left[1 - Q_2 + (Q_2 - Q_1)Cn^2\sqrt{\frac{Q_1 - Q_3}{6}\left|\frac{\beta}{C_x}\right|}\frac{\theta}{K_h}\right], \tag{36}$$

where Q_i (i = 1, 2, 3) are three single real roots (with $Q_1 > Q_2 > Q_3$) of the transcendental equation

$$\frac{Q^3}{6} - \frac{d^2Q^2}{2} - d^4Q + d^4(Q + d^2)\ln|Q + d^2| = H_0. \tag{37}$$

Hence, (36) is the expression of cnoidal waves, too. For $H_0 = a_2^3 / 3$ and with the initial phase satisfying $a_1 < Q$ ($\tau = 0$)$< Q_1$, there exists the solution of solitary waves in the form

$$\Phi = \frac{C_x}{\beta}(K_h^2 + n_1^2)\left(\frac{N}{n}\right)^2\left[1 - a_1 + (a_1 - Q_1)\sec^2 h\sqrt{\frac{Q_1 - a_1}{6}\left|\frac{\beta}{C_x}\right|}\frac{\theta}{K_h}\right], \tag{38}$$

in which Q_1 is the single root and a_1 the repeated root.

Since we have proved that the original issue has the periodic or solitary solution, which remains in the study with the aid of function approximation so that the solution of cnoidal or solitary waves can be viewed as the explicit solution of first approximation of the FAP and solitary waves.

It can be inferred from the explicit solution that effects upon wave behavior of zonal–shear flow, terrain and stratification are shown both in the constraints of wave formation (Section II) and in such parameters as wave velocity, length, amplitude and period in association with zonal shear basic flow, slope of terrain and stratification. Neglecting effects of the flow and terrain $(\bar{u} + C_h) = 0$, the exact expression of FAP waves is none other than that of cnoidal waves, which indicates the nonlinear characteristics of nonlinear waves and the effects on wave behavior of terrain, basic flow and stratification to full advantage.

VI. CONCLUDING REMARKS

Two basic constraints of the solutions of FAP and solitary waves in the barotropic and baroclinic atmospheres are obtained in terms of the geometric qualitative theory of phase plane of the dynamic system defined by the second–order nonlinear ordinary differential equation for the model under semi–geostrophic approximation. And the explicit approximation form of nonlinear waves is represented by cnoidal or solitary waves. The existence of periodic and solitary waves depends on initial phase disturbance, and effects of basic flow, terrain and stratification on wave behavior are displayed in the existence of waves and their mode (periodic and solitary). However, the conditions of nonlinear Rossby waves and the pattern, nature of the solution and representation with actual flow, stratification and terrain given are not dealt with here, which will be treated in a separate paper.

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