LONG VALID TIME ENERGY PERFECT CONSERVATIVE FIDELITY SPECTRAL SCHEMES OF BAROTROPIC PRIMITIVE EQUATIONS*

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ABSTRACT

In accordance with a new compensation principle of discrete computations, the traditional meteorological global (pseudo-) spectral schemes of barotropic primitive equation (s) are transformed into perfect energy conservative fidelity schemes, thus resolving the problems of both nonlinear computational instability and incomplete energy conservation, and raising the computational efficiency of the traditional schemes.

As the numerical tests of the new schemes demonstrate, in solving the problem of energy conservation in operational computations, the new schemes can eliminate the (nonlinear) computational instability and, to some extent even the (nonlinear) computational diverging as found in the traditional schemes. Further contrasts between new and traditional schemes also indicate that, in discrete operational computations, the new scheme in the case of nondivergence is capable of prolonging the valid integral time of the corresponding traditional scheme, and eliminating certain kind of systematical computational "climate drift", meanwhile increasing its computational accuracy and reducing its amount of computation. The working principle of this paper is also applicable to the problem concerning baroclinic primitive equations.

Key words: perfect energy conservative fidelity and traditional scheme, nonlinear computational instability and convergence, long valid time, computational efficiency, computational drift

I. INTRODUCTION

Nonlinear computational instability and convergence are two basic discrete computation problems which remain unsolved despite the many attempts made by the computational mathematicians and physicists. Retaining the same characteristics of the original continuous system is also one of the basic discrete computation problems for discrete system. Some recent work (Zeng and Zhang 1981; Wang and Ji 1990; Zhong 1992a; 1992b; 1992c; 1992d), however, has led to steady improvements even breakthroughs with regard to certain types of maths-physics problems' certain basic computational problems.

Some instant (weighted) square conservative schemes (Arakawa 1966; Lilly 1965; Chang 1977) have long been formulated abroad. In China, Zeng and Zhang (1981) provided

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an "instantly linearized" method for realizing Crank-Nicolson implicit complete square conservative scheme. Wang and Ji (1990) developed an explicit square conservative scheme formulation theory: to formulate an explicit square conservative scheme by adding a variable dissipation term, and realize a temporal-spatial difference complete square conservative explicit scheme. However, both the above designs are devoted to the same idea, namely, to transform control equation (s) of an evolution problem into a special form (symmetric operator form), then discretize it to formulate the scheme. This approach is somewhat limited, since technical difficulties may arise in practice, and the validity of the schemes is closely connected with the form in which the control equation (s) is discretized, moreover, not all important control equation (s) of evolution problem could be transformed into the special form, baroclinic primitive equation (s), for instance. Zhong (1991; 1992) has solved the formulation problem of a kind of perfect (quadratic) square conservative semi-implicit scheme and, in turn, constructed semi-implicit, explicit and (a kind of instantly linearized) implicit complete square conservative schemes' design suited for the control equation (s) of evolution problem in any form. More notably, Zhong has observed that the quadratic square conservative scheme formulated by the direct approach is capable of solving both nonlinear computational instability and nonlinear computational convergence, the two basic discrete computation problems mentioned above, and of constructing the economical computations as well (see Zhong 1992a; 1992c; 1992d). Moreover, a fidelity scheme concept is presented, and a general compensation formulation principle and approach of the fidelity scheme suited for maintaining any characteristic properties of evolution problems at any order accuracy of time difference are set up. Based on this compensation fidelity computation theory, the time difference-spatial (pseudo-) spectral expansion fidelity schemes of (weighted) square and nonsquare integral conservation property are respectively formulated and realized (Zhong 1992b).

Using the new computation principle of error inverse compensation (Zhong 1992b), this paper has improved a traditional meteorological (pseudo-) spectral scheme, realized an energy complete conservative (pseudo-) spectral scheme, solved corresponding nonlinear problems of computational instability and the problem of retaining energy conservation characteristic. This paper also tries to demonstrate the great potential of the type of energy complete conservative schemes in improving the computational effectiveness, prolonging the valid integral time and increasing the computation accuracy, in solving such problems as "climate drift" as well in constructing economical computations of the traditional schemes. The corresponding principle and approach used in this paper is also applicable to the problem of retaining other conservative characteristics of barotropic primitive equation (s) and the case of baroclinic primitive equation (s).

II. CONTROL EQUATION (S)

The barotropic primitive ("shallow water") equation (s) describing homogeneous incompressible geophysical fluid with free surface may be written as

$$\frac{d\boldsymbol{V}}{dt} = -f\boldsymbol{k} \times \boldsymbol{V} - \nabla \Phi, \tag{1a,b}$$

$$\frac{d\Phi}{dt} = -\Phi \nabla \cdot \boldsymbol{V}, \tag{1c}$$

where $\boldsymbol{V}$ is horizontal wind vector, u and v zonal and meridional wind subvector, Φ free surface geopotential height, f Coriolis parameter, $\boldsymbol{k}$ vertical unit vector, ∇ horizontal gradient operator, and $\mathrm{d}/\mathrm{d}t$ time derivative.

In the spherical coordinate, Eqs. (1a), (1b) and (1c) also can become

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{a\cos^2\varphi}\left\{\frac{\partial}{\partial\lambda}[U(\nabla^2\psi + f)] + \cos^2\varphi\frac{\partial}{\partial\mu}[V(\nabla^2\psi + f)]\right\}, \tag{2a}$$

$$\frac{\partial D}{\partial t} = \frac{1}{a\cos^2\varphi}\left\{\frac{\partial}{\partial\lambda}[V(\nabla^2\psi + f)] - \cos^2\varphi\frac{\partial}{\partial\mu}[U(\nabla^2\psi + f)]\right\} - \nabla^2\left[\frac{U^2 + V^2}{2(1-\mu^2)} + \Phi'\right], \tag{2b}$$

$$\frac{\partial \Phi'}{\partial t} = -\frac{1}{a\cos^2\varphi}\left\{\frac{\partial}{\partial\lambda}(U\Phi') + \cos^2\varphi\frac{\partial}{\partial\mu}(V\Phi')\right\} - \bar{\Phi}D, \tag{2c}$$

where λ, φ and t are longitude, latitude and time respectively; $\mu=\sin\varphi$; a the radius of the earth; ψ and χ streamfunction and velocity potential; $\zeta(=\nabla^2\psi)$ relative vorticity, $D=(\nabla^2\chi)$ divergence, $\bar{\Phi}$ global average geopotential independent of time, Φ' departure geopotential, $\Phi=\Phi'+\bar{\Phi}$; and scaled wind U, V as follows

$$U = -\frac{\cos^2\varphi}{a}\frac{\partial\psi}{\partial\mu} + \frac{1}{a}\frac{\partial\chi}{\partial\lambda}, \tag{3a}$$

$$V = \frac{1}{a}\frac{\partial\psi}{\partial\lambda} + \frac{\cos^2\varphi}{a}\frac{\partial\chi}{\partial\mu}. \tag{3b}$$

The enstrophy, energy, angular momentum and mass conservative characteristics of Eqs. (1a), (1b) and (1c) or (2a), (2b) and (2c) are the following:

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\frac{(\zeta+f)^2}{\Phi}4\pi a^2\cos\varphi\,\mathrm{d}\lambda\,\mathrm{d}\varphi = \text{constant}, \tag{4}$$

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left\{\frac{1}{2}(u^2+v^2)\Phi + \frac{1}{2}\Phi'^2\right\}4\pi a^2\cos\varphi\,\mathrm{d}\lambda\,\mathrm{d}\varphi = \text{constant}, \tag{5}$$

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}a\Phi(u\cos\varphi + \Omega\cos^2\varphi)4\pi a^2\cos\varphi\,\mathrm{d}\lambda\,\mathrm{d}\varphi = \text{constant}, \tag{6}$$

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\Phi' 4\pi a^2\cos\varphi\,\mathrm{d}\lambda\,\mathrm{d}\varphi = \text{constant}. \tag{7}$$

In the case of nondivergent motions ($D=0$), its control equation is degenerated as (2a), corresponding energy and mass conservative characteristics are

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\zeta^2 4\pi a^2\cos\varphi\,\mathrm{d}\lambda\,\mathrm{d}\varphi = \text{constant}, \tag{8}$$

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{2}(u^2+v^2)4\pi a^2\cos\varphi\,\mathrm{d}\lambda\,\mathrm{d}\varphi = \text{constant}, \tag{9}$$

and Rossby-Haurwitz waves are accurate solutions to Eq. (2a). They are also approximate solutions to the equations in the case of divergence.

The barotropic primitive equation (s) as a basic tool is capable of describing the significant atmospheric and oceanic motions, of which stratification is not a dominating factor (Pedlosky 1979).

III. TRADITIONAL METEOROLOGICAL (PSEUDO-) SPECTRAL SCHEMES

The barotropic primitive equation (s) traditional meteorological (pseudo-) spectral semi-implicit scheme (William 1972) is the following:

$$\frac{\zeta_{lm}^{n+1}-\zeta_{lm}^{n-1}}{2\Delta t}=F\zeta_{lm}^{n}, \tag{10a}$$

$$\frac{D_{lm}^{n+1}-D_{lm}^{n-1}}{2\Delta t}=FD_{lm}^{n}+\frac{l(l+1)}{2a^2}(\Phi'^{n+1}_{lm}+\Phi'^{n-1}_{lm}), \tag{10b}$$

$$\frac{\Phi'^{n+1}_{lm}-\Phi' D_{lm}^{n-1}}{2\Delta t}=F\Phi_{lm}^{n}-\bar{\Phi}\frac{D_{lm}^{n+1}+D_{lm}^{n-1}}{2}, \tag{10c}$$

where $F\zeta$, FD and $F\Phi$ are respectively the following nonlinear terms:

$$F\zeta=-\frac{1}{a\cos^2\varphi}\left\{\frac{\partial}{\partial\lambda}[U(\nabla^2\psi+f)]+\cos^2\varphi\frac{\partial}{\partial\mu}[V(\nabla^2\psi+f)]\right\}, \tag{11a}$$

$$FD=\frac{1}{a\cos^2\varphi}\left\{\frac{\partial}{\partial\lambda}[V(\nabla^2\psi+f)]-\cos^2\varphi\frac{\partial}{\partial\mu}[U(\nabla^2\psi+f)]\right\}$$
$$-\nabla^2\frac{U^2+V^2}{2(1-\mu^2)}, \tag{11b}$$

$$F\Phi=-\frac{1}{a\cos^2\varphi}\left\{\frac{\partial}{\partial\lambda}(U\Phi')+\cos^2\varphi\frac{\partial}{\partial\mu}(V\Phi')\right\}, \tag{11c}$$

and X_{lm}^{n} is a spectral expansion coefficient of variable X in terms of spherical harmonics $P_l^m(\mu)e^{im\lambda}$ at time n, and X can be any variable of ζ, D, Φ', $F\zeta$, FD or $F\Phi$. The spectral expansion coefficient of nonlinear term is determined by spectral transform method (Machenhauer 1981).

IV. ENERGY COMPLETE CONSERVATIVE (PSEUDO-) SPECTRAL SCHEMES

In accordance with the compensation principle, the energy conservative scheme for barotropic primitive equations of full discrete computations can be formulated as follows

$$\frac{\zeta_{lm}^{n+1}-\zeta_{lm}^{n}}{\Delta t}=F\zeta_{lm}^{n}+\varepsilon^n B\zeta_{lm}^{n}, \tag{12a}$$

$$\frac{D_{lm}^{n+1}-D_{lm}^{n}}{\Delta t}=FD_{lm}^{n}+\frac{l(l+1)}{a^2}\Phi'^{n+1}_{lm}+\varepsilon^n BD_{lm}^{n}, \tag{12b}$$

$$\frac{\Phi'^{n+1}_{lm}-\Phi' D_{lm}^{n}}{\Delta t}=F\Phi_{lm}^{n}-\bar{\Phi}D_{lm}^{n+1}+\varepsilon^n B\Phi_{lm}^{n}. \tag{12c}$$

Schemes (12a), (12b) and (12c) can also be written as

$$X_{lm}^{n+1}=\tilde{X}_{lm}+\varepsilon^n\tilde{\tilde{X}}_{lm}. \tag{13}$$

Compensation operator BX can be taken as

$$BX=\frac{X''-2X'+X}{2\Delta t}, \tag{14}$$

$\tilde{X}_{lm}$, $\tilde{\tilde{X}}_{lm}$ are determined by (12a), (12b) and (12c), X' and X'' are respectively the first and second step integral values of (12a), (12b) and (12c) while ε^n is considered as zero, Δt time step and X^n initial value. Here, $X=\{\zeta, D, \Phi'\}$.

In the case of divergent motions, towards retaining energy (weighted square) conservative characteristic, the compensation coefficient can be taken as

$$\varepsilon^{n}=\begin{cases}-\dfrac{b}{3a_{\varepsilon}}+\sqrt[3]{-\dfrac{q}{2}+\Delta^{1/2}}+\sqrt[3]{\dfrac{q}{2}-\Delta^{1/2}}, & \Delta>0\\ -\dfrac{b}{3a_{\varepsilon}}+2\left(-\dfrac{p}{3}\right)^{1/2}\cos\theta, & \\ \qquad \Delta\leqslant 0,\ |\cos\theta|\leqslant|\cos(\theta+120^{\circ})|,\ |\cos\theta|\leqslant|\cos(\theta+240^{\circ})| & \\ -\dfrac{b}{3a_{\varepsilon}}+2\left(-\dfrac{p}{3}\right)^{1/2}\cos(\theta+120^{\circ}), & \\ \qquad \Delta\leqslant 0,\ |\cos(\theta+120^{\circ})|\leqslant|\cos\theta|,\ |\cos(\theta+120^{\circ})|\leqslant|\cos(\theta+240^{\circ})| & \\ -\dfrac{b}{3a_{\varepsilon}}+2\left(-\dfrac{p}{3}\right)^{1/2}\cos(\theta+240^{\circ}), & \\ \qquad \Delta\leqslant 0,\ |\cos(\theta+240^{\circ})|\leqslant|\cos\theta|,\ |\cos(\theta+240^{\circ})|\leqslant|\cos(\theta+120^{\circ})| & \end{cases} \tag{15}$$

where

$$p=\frac{c}{a_{\varepsilon}}-\frac{b^{2}}{3a_{\varepsilon}^{2}}, \tag{16}$$

$$q=\frac{2b^{3}}{27a_{\varepsilon}^{3}}-\frac{bc}{3a_{\varepsilon}^{2}}+\frac{d}{a_{\varepsilon}}, \tag{17}$$

$$\Delta=\left(\frac{q}{2}\right)^{2}+\left(\frac{p}{3}\right)^{3}, \tag{18}$$

$$\theta=\frac{1}{3}\arccos\left\{-\frac{1}{2}q\left(-\frac{p}{3}\right)^{-3/2}\right\}, \tag{19}$$

$$a_{\varepsilon}=\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\frac{\tilde{U}_{i,j}^{2}+\tilde{V}_{i,j}^{2}}{2(1-\mu_{j}^{2})}\tilde{\tilde{\Phi}}'_{i,j}G_{j}^{(K_2)}, \tag{20}$$

$$b=\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\left\{\frac{\tilde{U}_{i,j}^{2}+\tilde{V}_{i,j}^{2}}{2(1-\mu_{j}^{2})}(\bar{\Phi}+\tilde{\Phi}'_{i,j})+\frac{\tilde{U}_{i,j}\tilde{\tilde{U}}_{i,j}+\tilde{V}_{i,j}\tilde{\tilde{V}}_{i,j}}{(1-\mu_{j}^{2})}\tilde{\tilde{\Phi}}'_{i,j}+\frac{1}{2}\tilde{\tilde{\Phi}}'^{2}_{i,j}\right\}G_{j}^{(K_2)}, \tag{21}$$

$$c=\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\left\{\frac{\tilde{U}_{i,j}\tilde{\tilde{U}}_{i,j}+\tilde{V}_{i,j}\tilde{\tilde{V}}_{i,j}}{1-\mu_{j}^{2}}(\bar{\Phi}+\tilde{\Phi}'_{i,j})+\frac{\tilde{\tilde{U}}_{i,j}^{2}+\tilde{\tilde{V}}_{i,j}^{2}}{2(1-\mu_{j}^{2})}\tilde{\tilde{\Phi}}'_{i,j}+\tilde{\tilde{\Phi}}'_{i,j}\tilde{\Phi}'_{i,j}\right\}G_{j}^{(K_2)}, \tag{22}$$

$$d=\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\left\{\frac{\tilde{U}_{i,j}^{2}+\tilde{V}_{i,j}^{2}}{2(1-\mu_{j}^{2})}(\bar{\Phi}+\tilde{\Phi}'_{i,j})+\frac{1}{2}\tilde{\Phi}'^{2}_{i,j}\right\}G_{j}^{(K_2)}-E_{0}, \tag{23}$$

$$E_{0}=\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}\left\{\frac{U_{0i,j}^{2}+V_{0i,j}^{2}}{2(1-\mu_{j}^{2})}(\bar{\Phi}+\Phi'_{0i,j})+\frac{1}{2}\Phi'^{2}_{0i,j}\right\}G_{j}^{(K_2)}, \tag{24}$$

$$x_{i,j}=\sum_{m=-m}^{m}\sum_{l=|m|}^{L(m)}x_{lm}p_{l}^{m}(\mu_{j})\mathrm{e}^{im\lambda_{i}}, \tag{25}$$

here $x_{i,j}=\tilde{\Phi}'_{i,j},\ \tilde{\tilde{\Phi}}'_{i,j}$, $x_{lm}=\tilde{\Phi}'_{lm},\ \tilde{\tilde{\Phi}}'_{lm}$;

$$\{Y_{i,j},\ Z_{i,j}\}=-a\sum_{m=-m}^{m}\sum_{l=|m|}^{L(m)}\left\{\frac{im}{l(l+1)}Y_{lm}P_{l}^{m}(\mu_{j})-\frac{1}{l(l+1)}(1-\mu_{j}^{2})Z_{lm}\frac{\mathrm{d}p_{l}^{m}(\mu_{j})}{\mathrm{d}\mu_{j}}\right.$$

$$\left.\frac{im}{l(l+1)}Z_{lm}P_{l}^{m}(\mu_{j})+\frac{1}{l(l+1)}(1-\mu_{j}^{2})\,Y_{lm}\frac{\mathrm{d}p_{l}^{m}(\mu_{j})}{\mathrm{d}\mu_{j}}\right\}\mathrm{e}^{im\lambda_{i}}, \tag{26}$$

here $\{Y_{i,j},\ Z_{i,j}\}=\{\tilde{U}_{i,j},\ \tilde{V}_{i,j}\}$, $\{\tilde{\tilde{U}}_{i,j},\ \tilde{\tilde{V}}_{i,j}\}$, $\{Y_{lm},\ Z_{lm}\}=\{\tilde{D}_{lm},\ \zeta_{lm}\}$, $\{\tilde{\tilde{D}}_{lm},\ \tilde{\tilde{\zeta}}_{lm}\}$. The physical meaning of Φ demands

$$\Phi=\bar{\Phi}+\Phi'\geqslant 0. \tag{27}$$

Thereby, instead of Φ, its absolute value can be taken in practice. It can be easily proved that scheme (12) is now a weighted square energy complete conservative scheme.

In the case of nondivergent motions, toward retaining energy (square) conservative characteristic, the compensation coefficient can be taken as

$$\varepsilon^n = -\frac{C_2}{C_1}\left[1-\sqrt{1-\frac{C_3C_1}{C_2^2}}\right], \tag{28}$$

$$C_1 = \sum_{i=1}^{K_1}\sum_{j=1}^{K_2} G_j^{(K_2)} \frac{\widetilde{U}_{i,j}^2+\widetilde{V}_{i,j}^2}{2(1-\mu_j^2)}, \tag{29}$$

$$C_2 = \sum_{i=1}^{K_1}\sum_{j=1}^{K_2} G_j^{(K_2)} \frac{\widetilde{U}_{i,j}\widetilde{U}_{i,j}+\widetilde{V}_{i,j}\widetilde{V}_{i,j}}{2(1-\mu_j^2)}, \tag{30}$$

$$C_3 = \sum_{i=1}^{K_1}\sum_{j=1}^{K_2} G_j^{(K_2)} \frac{\widetilde{U}_{i,j}^2+\widetilde{V}_{i,j}^2-U_{0i,j}^2-V_{0i,j}^2}{2(1-\mu_j^2)}, \tag{31}$$

$$G_j^{(K_2)} = \frac{2(1-\mu_j^2)}{[K_2P_{K_2-1}(\mu_j)]^2}. \tag{32}$$

In all the formulas in this section, $G_j^{(K_2)}$ is referred to as Gaussian weighted factor, $X_{0i,j}$ as initial value of variable X at grid point (i,j), $X=\{U, V, \Phi'\}$. It can be easily proved that Scheme (12) is a square energy complete conservative fidelity scheme.

It is clear that as a weighted square or a square energy complete conservative fidelity scheme, Scheme (12) is a stable scheme capable of completely solving both linear and nonlinear computational instability problems.

V. NUMERICAL TEST

Since Phillips' s work was presented in 1959, Rossby-Haurwitz wave test has become a routine test for barotropic primitive equations scheme. This is mainly because Rossby-Haurwitz wave is not only meteorologically significant, but also an approximate solution in the case of divergent motions and an accurate solution to nondivergent motions of the nonlinear equations.

Physically initial conditions of Rossby-Haurwitz wave test here are taken as follows. Nondivergent initial velocity ($D=0$) is determined by streamfunction (see Fig. 1):

$$\psi_0 = -a^2A_0\sin\varphi + a^2A_1\cos^{m_0}\varphi\sin\varphi\cos m_0\lambda, \tag{33}$$

and initial geopotential Φ_0 has the form

$$\Phi_0 = \bar{\Phi} + a^2A(\varphi) + a^2B(\varphi)\cos m_0\lambda + a^2C(\varphi)\cos 2m_0\lambda, \tag{34}$$

$$A(\varphi) = \frac{1}{2}A_0(2\Omega+A_0)\cos^2\varphi + \frac{1}{4}A_1^2\cos^{2m_0}\varphi\,[(m_0+1)\cos^2\varphi + (2m_0^2-m_0-2) - 2m_0^2\cos^{-2}\varphi], \tag{35}$$

$$B(\varphi) = \frac{2(\Omega+A_0)A_1}{(m_0+1)(m_0+2)}\cos^{m_0}\varphi\,[(m_0^2+2m_0+2)-(m_0+1)^2\cos^2\varphi], \tag{36}$$

$$C(\varphi) = \frac{1}{4}A_1^2\cos^{2m_0}\varphi[(m_0+1)\cos^2\varphi-(m_0+2)]. \tag{37}$$

Operationally, the truncation of spherical harmonics is trapezoidal, with maximum truncated number of 11; the number of regularly-spaced longitudinal grid points is 32, and 26 of Gaussian latitudinal grid points. Other parameters are taken as $A_0 = A_1 = 3.924\times10^{-6}\mathrm{s}^{-1}$, $m_0 = 4$, $a=6371\times10^6$ m, $\bar{\Phi} = 7.84\times10^4$ m, $\Omega=7.29\times10^{-5}\mathrm{s}^{-1}$. A computational initial condition of traditional scheme (10) is produced by a commonly used half time step integration, i. e., let computational initial values at time $t=\Delta t/2$ be equal to the values at time $t=0$, forward integrate a step with time step value $\Delta t/2$, then forward calculate routinely with time step value Δt. Besides, the schemes are run on computer Convex-1 Institute of Atmospheric Physics. The codes employ single precision arithmetic (7 significant digits), although the requisite associated legendre polynomials are generated in double precision.

Contrast test between the traditional scheme (10a—c) and the new conservative scheme (12a—c) in an ordinary (divergent) case with physically initial conditions (33—34) demonstrates that in the traditional scheme, there is a critical time step (about 100 min.); when time step is shorter than the critical value, the computational relative error of conservative energy, enstrophy and angular momentum global integration all vary in a way of the three step regularity: oscillation variation — symmetric oscillation increasing — rapid asymmetric increasing, which can not be altered by reducing time step value (test margin: time step $\geqslant$ 10 min., though conservative mass global integration is retained, and its stable integral time is less than 190 days (see Fig. 2); when time step is greater than this critical value, the error of the conservative integrations increase rapidly (nonlinear computation instability), its stable integration time grows much shorter than 190 days. Namely, there arises a type of systematical "climate (computational) drift".

For the new energy weighted square conservative semi-implicit fidelity scheme, its energy as well as mass conservative characteristic is well retained (see Figs. 3a and 3b), its stable integration time is much longer than 190 days, and its stable integral step is greater than the critical value (100 min.) of the traditional scheme. Note, for instance, its stable integral time can reach 800 or more days at integral time step 6 hours. Besides, in the same test, the maximum value of stable integral step of an explicit conservative scheme can merely reach approximately 15 min. This shows that the semi-implicit conservative scheme is capable of greatly reducing the overall amount of computation as compared with the explicit conservative scheme.

For the divergent problems, although the basic performance of the scheme could be

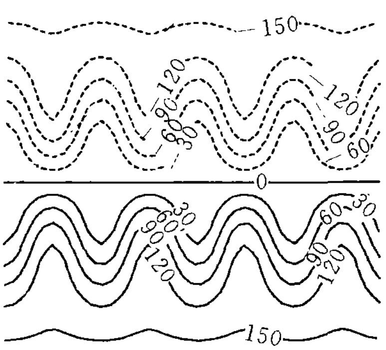

Fig. 1. Initial stream function of Rossby-Haurwitz wave.

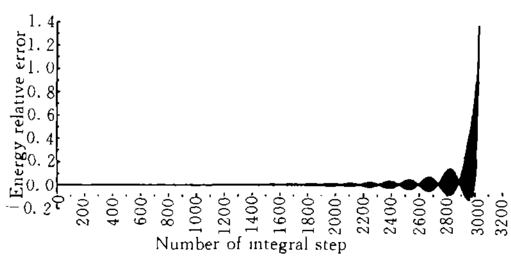

Fig. 2. The computational variation of conservative energy global integration (traditional semi-implicit (pseudo-) spectral scheme of divergent barotropic primitive equation, integral step 90 min., integral time 189 d).

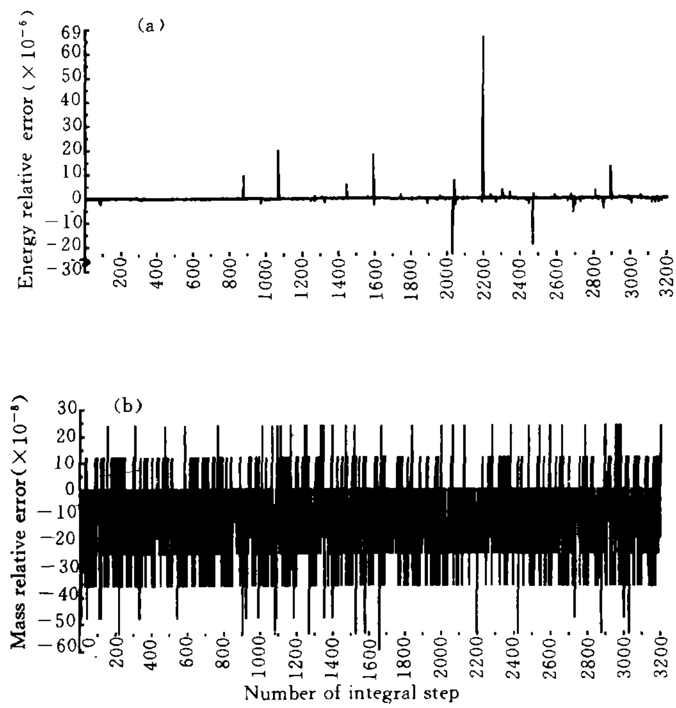

Fig. 3. The computational variation of conservative global integration of energy (a) and mass (b) (energy conservative fidelity semi-implicit (pseudo-) spectral scheme of divergent barotropic primitive equation, integral step 6 h, integral time 800 d, single precision arithmetic).

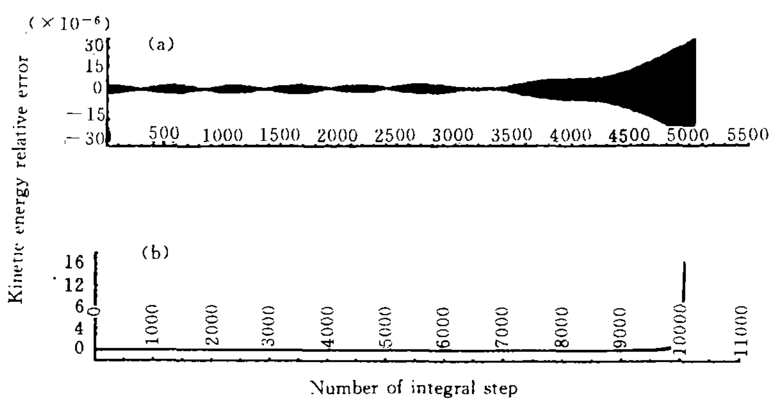

Fig. 4. The computational variation of conservative kinetic energy global integration (traditional explicit (pseudo-) spectral scheme of nondivergent barotropic primitive equation, integral step 10 min., integral time (a) 35 d; (b) 70 d).

checked to some extent, they could not be determined objectively and accurately by the presence or absence of metamorphosis of the wave for stable numerical integrations, as Rossby-Haurwitz wave (33) is merely an approximate solution. Further contrast test between traditional meteorological and new type energy conservative scheme (10a) and (12a) of nondivergent barotropic primitive equation or vorticity equation is conducted, since Rossby-Haurwitz

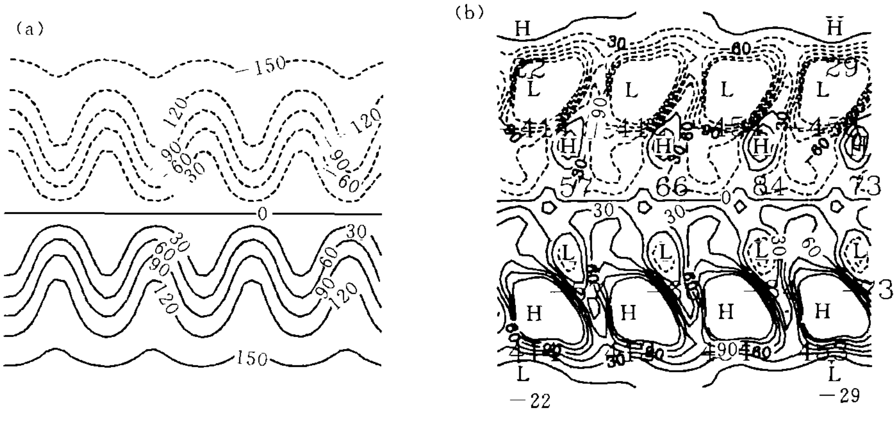

Fig. 5. The computed streamfunction of Rossby-Haurwitz wave (traditional (pseudo-) spectral scheme of nondivergent barotropic primitive equation. integral step 10 min.. integral time: (a) 60 d; (b) 70 d).

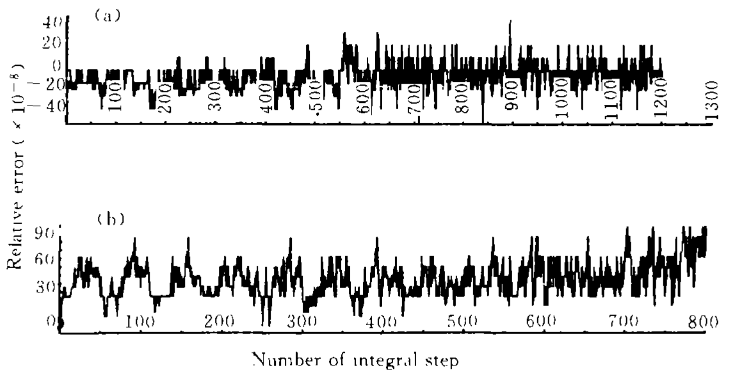

Fig. 6. The computational variations of conservative global integration of kinetic energy (a) and enstrophy (b) (kinetic energy conservative fidelity explicit (pseudo-) spectral scheme of nondivergent barotropic primitive equation. single precision arithmetic. integral step 3 h. integral time: (a) 150 d; (b) 100 d).

wave (33) is now its accurate solution, and the test could therefore be more objective and clear. The test has demonstrated that, similar to the divergent case of the traditional scheme, there is a critical time step (about 320 min.); when time step is shorter than this critical value, the computational relative error of conservative energy and enstrophy global integration both display a three step variation regularity: oscillation variation-symmetric oscillation increasing-rapid asymmetric increasing (see Table 1 and Figs. 4a and 4b), and the duration for nearly retaining the wave pattern is approximately 60 days (<70 days) (see Figs. 5a, 5b). They can not be altered by reducing time step value. When time step is greater than this critical value, the error of the conservative integrations increase rapidly (nonlinear computation instability), its stable integration time is much shorter than 60 days. There also exists the same kind of systematical "climate (computational) drift".

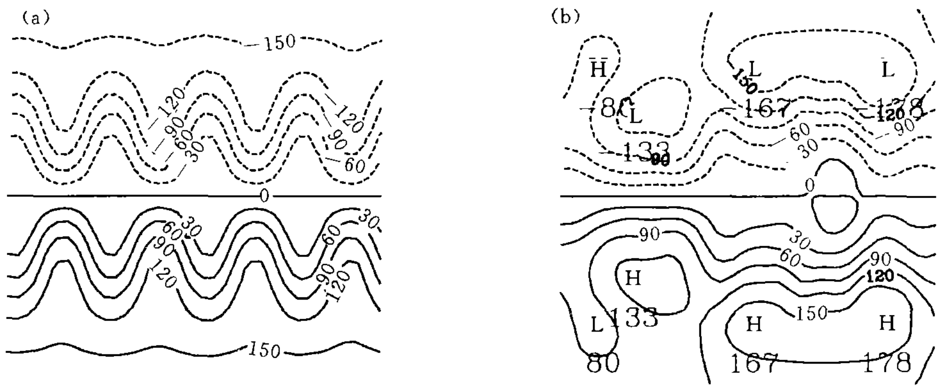

Fig. 7. The computed stream function of Rossby-Haurwitz wave (kinetic energy conservative fidelity (pseudo-) spectral scheme of nondivergent barotropic primitive equation, integral step 3 hours, integral time: (a) 90 d; (b) 150 d).

Table 1. The Computed Variations of the Conservative Kinetic Energy and Enstrophy Global Integrations (The Traditional (Pseudo-) Spectral Scheme of Nondivergent Barotropic Primitive Equation, Integral Step 10 min., Single Precision Arithmetic)

Day	Kinetic energy (m^2/s)	Enstrophy (m^2/s)	Day	Kinetic energy (m^2/s)	Enstrophy (m^2/s)
0	0.1220760 E+5	0.8847784 E−8	30	0.1220771 E+5	0.8847945 E−8
35	0.1220796 E+5	0.8848321 E−8	40	0.1220877 E+5	0.8849525 E−8
45	0.1221150 E+5	0.8853549 E−8	50	0.1222102 E+5	0.8867624 E−8
55	0.1225634 E+5	0.8919797 E−8	60	0.1241432 E+5	0.9152241 E−8
65	0.1348819 E+5	0.1071767 E−8	70	0.2136956 E+5	0.9839725 E−8
>70	overflow	overflow			

For the new kinetic energy square conservative explicit fidelity scheme, except for computer round-off error, both its energy and enstrophy conservative characteristic are completely retained (see Table 2 and Figs. 6a, 6b), thereby the problem of nonlinear computational instability is eliminated. Also solved is the problem of nonlinear computational convergence within the integral time (90 days) much longer than the useful integral time of the traditional scheme (compare Figs. 5a, 5b and Fig. 7a), thereby the "climate drift" problem is solved as well (see and compare Fig. 1, Figs. 6a, 6b, 7a, 7b and Figs. 4a, 4b, 5a, 5b; Tables 2 and 1). However, there is no equivalent theorem of nonlinear computational stability and convergence (see Fig. 6a and Fig. 7b). In contrast with the traditional scheme, the kinetic energy conservative fidelity scheme is capable of greatly prolonging its valid integral time and in the same time, greatly increasing its accuracy and reducing its amount of computations (compare Table 1, Figs. 4—5 and Table 2, Figs. 6—7). The convergent valid time step of the fidelity scheme can reach as long as 57 hours.

Table 2. The Computed Variations of the Conservative Kinetic Energy and Enstrophy Global Integrations (A Kinetic Energy Conservative Fidelity (Pseudo-) Spectral Scheme of Nondivergent Barotropic Primitive Equation. Integral Step 3 h. Single Precision Arithmetic)

Day	Kinetic energy (m^2/s)	Enstrophy (m^2/s)	Day	Kinetic energy (m^2/s)	Enstrophy (m^2/s)
0	0.12207602 E+5	0.8847784 E−8	30	0.12207600 E+5	0.8847788 E−8
35	0.12207602 E+5	0.8847789 E−8	40	0.12207602 E+5	0.8847784 E−8
45	0.12207601 E+5	0.8847786 E−8	50	0.12207601 E+5	0.8847789 E−8
55	0.12207602 E+5	0.8847784 E−8	60	0.12207604 E+5	0.8847788 E−8
65	0.12207602 E+5	0.8847786 E−8	70	0.12207604 E+5	0.8847784 E−8
75	0.12207600 E+5	0.8847784 E−8	80	0.12207601 E+5	0.8847789 E−8
85	0.12207601 E+5	0.8847786 E−8	90	0.12207604 E+5	0.8847788 E−8
95	0.12207601 E+5	0.8847791 E−8	100	0.12207600 E+5	0.8847792 E−8

V. SUMMARY AND DISCUSSION

The sources for all the basic problems of discrete computations can be attributed to the emerging of errors of discrete computation. If we are able to completely eliminate the error according to its sources and the way it is introduced, then we are able to calculate exactly. The present work is to, based on the principle of retaining the conservative characteristic (s) of the original continuous problem, formulate a new type of energy complete conservative scheme by means of compensating thereby eliminating the discrete computational errors averagely and accordingly at each computational grid (component) of the corresponding traditional barotropic primitive equation (s) spectral scheme in accordance with the sources of the errors and the way they emerge. The new schemes thereby possess better performance than the traditional ones, and such improvements could be of fundamental significance. Further numerical contrast test between the new energy conservative and traditional schemes related to the Rossby-Haurwitz wave, the accurate solution, has also confirmed this inference. For instance, as the contrast test reveals, there is a kind of systematic error existing in the traditional scheme which leads to a type of "climate drift" phenomena incapable of being eliminated by reducing time step value. The new scheme can prolong its useful integral time by more than 50 percent with 18 times its original time step (averagely, about one-ninth its amount of computation), and make it possible for the computational solution converging to the physical one with more computer round-off errors. All these mean that, in contrast with the traditional scheme, the new energy complete conservative fidelity scheme does have higher computational accuracy, longer valid integral time, and greater computational efficiency. The new scheme design has theoretically and operationally solved both nonlinear computational instability and the problem of retaining energy conservative characteristic. It has also provided an operational case for resolving nonlinear computational diverging as found in traditional scheme to some extent.

It needs to be pointed out that time discrete errors are conventionally regarded as trivial (Jiang et al. 1989) and yet, as revealed in this work and other studies (Wang and Ji 1990; Zhong 1992a; 1992c; 1992b), many significant improvements in the quantity and quality of the performance of the scheme are exactly obtained by reasonably eliminating systematic errors appearing in time discrete computation. It therefore becomes evident that this conventional view does not universally hold true, and reconsideration is required.

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