THE ADJUSTMENT OF WIND TO EKMAN FLOW WITHIN THE PLANETARY BOUNDARY LAYER

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ABSTRACT

By using simple barotropic boundary layer equations with constant eddy viscosity, the analytical solution is obtained under the initial condition that the distribution of wind for a given pressure is not the well-known Ekman flow. We have found that the wind will finally adjust to the Ekman flow at a rate faster than that of geostrophic adjustment. We have also found that the thinner the boundary layer, the faster the rate of adjustment.

In the free atmosphere, the wind field is basically in geostrophic balance with the pressure field. However, in some local regions, the geostrophic departure may exist. It will be dispersed through the effect of gravity-inertial waves, and finally, the geostrophic balance can be established again. A lot of work has been done on geostrophic adjustment. Similar to the feature of the quasi-geostrophic flow in the free atmosphere, the wind within the boundary layer is mainly expressed by the Ekman flow, which is a result of balance between the pressure gradient force, the Coriolis force, and the viscous force. In the boundary layer, there exists the problem like that of geostrophic adjustment. For example, if the initial wind field is not in the form of Ekman flow, what will happen? By what effects will the Ekman balance be established? Up to now, little discussions have been made on these questions. In this paper, we will preliminarily analyze this adjustment process using the simplest model.

I. PROBLEM

A set of linearized equations which describe the horizontal motion in the planetary boundary layer can be written as

$$\frac{\partial u}{\partial t}=fv-\frac{1}{\rho}\frac{\partial p}{\partial x}+k\frac{\partial^2 u}{\partial z^2}, \tag{1}$$

$$\frac{\partial v}{\partial t}=-fu-\frac{1}{\rho}\frac{\partial p}{\partial y}+k\frac{\partial^2 v}{\partial z^2}, \tag{2}$$

where u, v are the velocity components along x, y directions, respectively; f is the Coriolis parameter; p pressure; ρ density; and k the coefficient of turbulent eddy viscosity. In this paper, we consider f, ρ and k as constants.

We assume that the boundary layer is barotropic, therefore the pressure gradient force is independent of height throughout the boundary layer. By using u_g, v_g to denote components of the geostrophic wind in the x, y directions, respectively, then Eqs. (1) and (2) may be rewritten as

$$\frac{\partial u}{\partial t}=f(v-v_g)+k\frac{\partial^2 u}{\partial z^2}, \tag{3}$$

$$\frac{\partial v}{\partial t}=-f(u-u_g)+k\frac{\partial^2 v}{\partial z^2}. \tag{4}$$

The initial conditions for Eqs. (3) and (4) are

$$t=0,\quad u=ug(1-e^{-\alpha z}), \tag{5}$$

$$v=v_g(1-e^{-\alpha z}), \tag{6}$$

where α is a parameter. The wind field expressed by Eqs. (5) and (6) has two features as follows:

(1) As $z=0$, then $u=v=0$; (7)

as $z\to\infty$, then $u\to u_g$, $v\to v_g$. (8)

Eq. (7) expresses that the fluid at the lower boundary is non-slipped and Eq. (8) expresses that the wind velocity at the top of the boundary layer must approach its geostrophic value. We choose Eqs. (7) and (8) as the boundary conditions for Eqs. (3) and (4).

(2) As $\alpha=\sqrt{\frac{f}{2k}}(1+i)$, then distribution of wind velocity expressed by Eqs. (5) and (6) is in agreement with that of classical Ekman flow. As $\alpha\neq\sqrt{\frac{f}{2k}}(1+i)$, for example, α is a positive real number, the initial wind field differs from the Ekman flow. Our problem is to study how the wind field, which is not in agreement with the Ekman flow, will change.

II. THE STEADY STATE MOTION

At first, we will qualitatively investigate what kind of characteristic motion the non-Ekman flow will reach eventually.

It is conventional to designate the geostrophic wind u_g, v_g as

$$\frac{\partial \pi}{\partial x}=fv_g,\qquad \frac{\partial \pi}{\partial y}=-fu_g, \tag{9}$$

where π denotes the geopotential height from mean value, which is unchanged with height because of the initial assumption that the boundary layer is barotropic. Consequently, from Eqs. (3) and (4) we may get the vorticity equation and the divergence equation:

$$\frac{\partial \zeta}{\partial t}=-fD+k\frac{\partial^2 \zeta}{\partial z^2}, \tag{10}$$

$$\frac{\partial D}{\partial t}=f\zeta-\nabla^2\pi+k\frac{\partial^2 D}{\partial z^2}, \tag{11}$$

where ζ and D are relative vorticity and divergence, respectively, namely,

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},\qquad D=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}. \tag{12}$$

Introducing streamfunction Ψ and velocity potential function φ, we have

$$u=-\frac{\partial \Psi}{\partial y}+\frac{\partial \varphi}{\partial x},\quad v=\frac{\partial \Psi}{\partial x}+\frac{\partial \varphi}{\partial y}, \tag{13}$$

and

$$\zeta=\nabla^2\Psi, \qquad D=\nabla^2\varphi. \tag{14}$$

Substituting Eq. (14) into Eqs. (10) and (11) yields

$$\nabla^2\left(\frac{\partial\Psi}{\partial t}+f\varphi-k\frac{\partial^2\Psi}{\partial z^2}\right)=0. \tag{15}$$

$$\nabla^2\left(\frac{\partial\varphi}{\partial t}-f\Psi+\pi-k\frac{\partial^2\varphi}{\partial z^2}\right)=0. \tag{16}$$

For a wave-type disturbance the horizontal Laplacian operator is proportional to the modulus of the wavenumber vector. Therefore, ∇^2 may be removed and Eqs. (15) and (16) may be written as:

$$\frac{\partial\Psi}{\partial t}+f\varphi-k\frac{\partial^2\Psi}{\partial z^2}=0, \tag{17}$$

$$\frac{\partial\varphi}{\partial t}-f\Psi+\pi-k\frac{\partial^2\Psi}{\partial z^2}=0. \tag{18}$$

Consequently, Eqs. (5)—(6) are reduced to

$$t=0, \varphi=0, \Psi=\frac{\pi}{f}(1-e^{-az}), \tag{19}$$

$$z=0,\ \ \varphi=0,\ \ \Psi=0, \tag{20}$$

$$z\to\infty,\ \ \varphi=0,\ \ \Psi=\frac{\pi}{f}. \tag{21}$$

From Eqs. (17) and (18), we can obtain the steady solution $\bar{\Psi}$, $\bar{\varphi}$, which satisfy

$$f\bar{\varphi}-k\frac{\partial^2\bar{\Psi}}{\partial z^2}=0, \tag{22}$$

$$f\bar{\Psi}-\bar{\pi}+k\frac{\partial^2\bar{\varphi}}{\partial z^2}=0. \tag{23}$$

This is the steady solution or the solution in equilibrium state, in fact, just the classical Ekman solution.

Elimination of $\bar{\varphi}$ from Eqs. (22) and (23) yields

$$\frac{\partial^4\bar{\Psi}}{\partial z^4}+\frac{f^2}{k^2}\bar{\Psi}=\frac{f}{k^2}\bar{\pi}. \tag{24}$$

Solving the equation with the boundary conditions

$$z=0,\ \ \bar{\Psi}=0,\ \ \bar{\varphi}=0, \tag{25}$$

$$z\to\infty,\ \ \bar{\varphi}=0,\ \ \bar{\Psi}=\frac{\bar{\pi}}{f}, \tag{26}$$

we get the solution of Eq. (24) as follows

$$\bar{\Psi}=-\frac{\bar{\pi}}{2f}\left(e^{-\sqrt{\frac{fi}{k}}z}+e^{\sqrt{-\frac{fi}{k}}z}-2\right); \tag{27}$$

and from Eq. (22), we have

$$\bar{\varphi}=-\frac{i\bar{\pi}}{2f}\left(e^{-\sqrt{\frac{fi}{k}}z}-e^{\sqrt{-\frac{fi}{k}}z}\right). \tag{28}$$

Thus, for a given $\bar{\pi}$ field, $\bar{\Psi}$ and $\bar{\varphi}$ field can be obtained. Using Eq. (13), we can get wind field as

$$\bar{u}+i\bar{v}=(u_g+iv_g)\left(1-e^{-\sqrt{\frac{f}{2k}}(1+i)z}\right), \tag{29}$$

this is the classical Ekman flow.

III. VARIATIONS OF THE PART OF NON-EQUILIBRIUM

Suppose π is independent of height z as well as time t, and set

$$w=\Psi-\frac{\pi}{f}+i\varphi, \tag{30}$$

then Eqs. (17) and (18) can be combined in

$$\frac{\partial w}{\partial t}=ifw+k\frac{\partial^2 w}{\partial z^2}. \tag{31}$$

The initial and boundary conditions of Eqs. (19)—(21) now can be written as

$$t=0, \qquad w=-\frac{\pi}{f}e^{-az}, \tag{32}$$

$$z=0, \qquad w=-\frac{\pi}{f}, \tag{33}$$

$$z\to\infty, \quad w=0. \tag{34}$$

With Laplacian transformation, we can obtain

$$\begin{aligned} w=-\frac{\pi}{2f}\Big[e^{-\sqrt{\frac{f}{2k}}(1-i)z}\operatorname{erfc}\left(\frac{z}{\sqrt{4kt}}-\sqrt{\frac{ft}{2}}+i\sqrt{\frac{ft}{2}}\right)+e^{\sqrt{\frac{f}{2k}}(1-i)z} \\ \times\operatorname{erfc}\left(\frac{z}{\sqrt{4kt}}+\sqrt{\frac{ft}{2}}-i\sqrt{\frac{ft}{2}}\right)\Big]+\frac{\pi}{2f}e^{ift}e^{ka^2 t}\Big\{e^{az}\operatorname{erfc}\left(\frac{z}{\sqrt{4kt}}+\sqrt{ka^2t}\right)+e^{-az} \\ \times\Big[\operatorname{erfc}\left(\frac{z}{\sqrt{4kt}}-\sqrt{ka^2t}\right)-2\Big]\Big\}. \end{aligned} \tag{35}$$

As $t\to\infty$, Eq. (35) tends to

$$W\to-\frac{\pi}{f}e^{-\sqrt{\frac{f}{2k}}(1-i)z}. \tag{36}$$

On the other hand, substitution of Eqs. (27) and (28) into (3) yields the steady state of w as

$$\bar{w}=\bar{\Psi}-\frac{\bar{\pi}}{f}+i\bar{\varphi}=-\frac{\bar{\pi}}{f}e^{-\sqrt{\frac{f}{2k}}(1-i)z}. \tag{37}$$

By comparing Eqs. (36) with (37), it is easy to find that the steady solution Eq. (35) is just the Ekman flow. We arrive, through analytical method, at the same conclusion that the steady state of motion is Ekman flow.

Next we will investigate the change of the non-steady state component.

In terms of an approximate formulation

$$\operatorname{erfc}(x+iy)=1-\operatorname{erf}(x+iy)\approx 1-\left[\operatorname{erf}(x)+\frac{1}{2\pi x}e^{-x^2}(1-\cos 2xy+i\sin 2xy)\right], \tag{38}$$

we can approximately obtain

$$\frac{2f}{\pi}w=-e^{-\sqrt{\frac{f}{2k}}z}\left(\cos\sqrt{\frac{f}{2k}}z+i\sin\sqrt{\frac{f}{2k}}z\right)\left\{1-\mathrm{erf}\left(\frac{z}{\sqrt{4kt}}-\sqrt{\frac{ft}{2}}\right)\right.$$

$$-\frac{1}{2\pi\left(\frac{z}{\sqrt{4kt}}-\sqrt{\frac{ft}{2}}\right)}e^{-\left(\frac{z^2}{4kt}-\sqrt{\frac{f}{2k}}z+\frac{ft}{2}\right)}\left.\left[1-\cos\left(\sqrt{\frac{f}{2k}}z-ft\right)+i\sin\left(\sqrt{\frac{f}{2k}}z-ft\right)\right]\right\}$$

$$+e^{\sqrt{\frac{f}{2k}}z}\left(\cos\sqrt{\frac{f}{2k}}z-i\sin\sqrt{\frac{f}{2k}}z\right)\left\{1-\mathrm{erf}\left(\frac{z}{\sqrt{4kt}}+\sqrt{\frac{ft}{2}}\right)\right.$$

$$-\frac{1}{2\pi\left(\frac{z}{\sqrt{4kt}}+\sqrt{\frac{ft}{2}}\right)}e^{-\left(\frac{z^2}{4kt}+\sqrt{\frac{f}{2k}}z+\frac{ft}{2}\right)}\left.\left[1-\cos\left(\sqrt{\frac{f}{2k}}z+ft\right)+i\sin\sqrt{\frac{f}{2k}}z+ft\right)\right]\right\}$$

$$+(\cos ft+i\sin ft)e^{ka^2t}\left\{e^{az}\left[1-\mathrm{erf}\left(\frac{z}{\sqrt{4kt}}+\sqrt{ka^2t}\right)\right]-e^{-az}\left[1+\mathrm{erf}\left(\frac{z}{\sqrt{4kt}}-\sqrt{ka^2t}\right)\right]\right\}. \tag{39}$$

Let D_1 be the divergence part of the non-steady state

$$D_1=\frac{2f}{\pi}(\varphi-\bar{\varphi}), \tag{40}$$

and the real form of Eq. (28) is written as

$$\bar{\varphi}=-\frac{\pi}{f}e^{-\sqrt{\frac{f}{2k}}z}\sin\sqrt{\frac{f}{2k}}z. \tag{41}$$

Since φ is the imaginary part of w, with these expressions, then we have

$$D_1=\sin ft\left\{\frac{-2z\sqrt{kt}}{\pi(z^2-2kft^2)}e^{-\left(\frac{z^2}{4kt}+\frac{ft}{2}\right)}+e^{ka^2t}\left[e^{az}\left(1-\mathrm{erf}\left(\frac{z}{\sqrt{4kt}}+\sqrt{ka^2t}\right)\right)\right.\right.$$

$$\left.\left.-e^{-az}\left(1+\mathrm{erf}\left(\frac{z}{\sqrt{4kt}}-\sqrt{ka^2t}\right)\right)\right]\right\}+\sin\sqrt{\frac{f}{2k}}z\left\{\frac{2z\sqrt{kt}}{\pi(z^2-2kft^2)}e^{-\left(\frac{z^2}{4kt}+\frac{ft}{2}\right)}\right.$$

$$\left.+e^{-\sqrt{\frac{f}{2k}}z}\left[1+\mathrm{erf}\left(\frac{z}{\sqrt{4kt}}-\sqrt{\frac{ft}{2}}\right)\right]-e^{\sqrt{\frac{f}{2k}}z}\left[1-\mathrm{erf}\left(\frac{z}{\sqrt{4kt}}+\sqrt{\frac{ft}{2}}\right)\right]\right\}. \tag{42}$$

The first term on the right hand side expresses the inertial oscillation with frequency f, and its amplitude tends to zero with t increasing. So does the second term. Since these two terms involve the frequency of the inertial oscillation f and the coefficient of turbulence eddy viscosity k, it is evident that the part of non-Ekman balance tends to be trivial under the influence of dispersion of the inertial oscillation and dissipation of the turbulence viscosity.

If take $z=\pi\sqrt{\frac{2k}{f}}$, then the second term of Eq. (42) tends to zero. If t is large enough, we have

$$D_1\approx-\sqrt{\frac{2\pi}{f}}\frac{1}{a^2kt^{3/2}}\sin ft. \tag{43}$$

IV. CONCLUSIONS

Comparing Eq. (43) with the example of geostrophic adjustment, the following conclusion remarks can be obtained.

First, we find that the part of non-Ekman balance tends ultimately to zero under the influence of the dispersion of the inertial oscillation and the dissipation of the turbulent viscosity, and the wind velocity always adjusts to Ekman flow.

In the case of geostrophic adjustment, there is an approximate expression (Wu, 1983)

$$\varphi(0,0,t) \sim \frac{1}{t}\cos ft, \tag{44}$$

where φ (0,0,t) is the potential function of wind at the center (0,0) of a region where the intense ageostrophic component occurs. Furthermore, from Eq. (43), we can get

$$D_1 \sim \frac{1}{t^{3/2}}\sin ft. \tag{45}$$

Comparing the powers of t in the denominators in Eqs. (44) and (45), we obtain the second conclusion that the Ekman adjustment process is faster than geostrophic adjustment process.

In addition, from Eqs. (5) and (6), we can find that the larger the α, the lower is the height where u, v equal u_g, v_g respectively, meaning that the thinner the depth of the boundary layer will be. Since the wind velocity at this height approaches geostrophic wind, the relation between wind and pressure fields hold the geostrophic balance. Thus the effect of the turbulent viscosity is so small that it may be neglected. From Eq. (43) we see that the larger the α, the smaller the $|D_1|$, i.e. the smaller the t at which D_1 is equal to minimum. Therefore, our third conclusion is that the thinner the boundary layer, the faster the rate of Ekman adjustment process is.

Owing to the assumption that the pressure field does not change with time, the interaction between the free atmosphere and boundary layer can not be considered. The further study of coupling these two layers will be meaningful.

REFERENCE

Wu Rongsheng et al. (1983), *Dynamic Meteorology*, Sci. & Tech. Press, Shanghai (in Chinese).