# AN IMPROVED SEMI-IMPLICIT TIME DIFFERENCE SCHEME OF SPECTRAL MODEL AND NUMERICAL APPLICATIONS*

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ABSTRACT

In fact, the popular semi-implicit time difference scheme of spectral model still includes some important linear terms using time explicit difference scheme, and the major terms are directly related to fast internal- and external-gravity waves in the atmospheric forecasting equation. Additionally, due to using time difference on two terms at different time, the popular scheme artificially introduces unbalance between pressure-gradient force and Coriolis force terms while numerically computing their small difference between large quantities. According to the computational stability analysis conducted to the linear term time difference scheme in simple harmonic motion equation, one improved semi-implicit time difference scheme is also designed in our study. By adopting a kind of revised time-explicit-difference scheme to these linear terms that still included in spectral model governing equations, the defect of spectral model which only partly using semi-implicit integrating scheme can be overcome effectively. Moreover, besides all spectral coefficients of prognostic equations, especially of Helmholtz divergence equation, can be worked out without any numerical iteration, the time-step (computation stability) can also be enlarged (enhanced) by properly introducing an adjustable coefficient.

**Key words**: spectral model, semi-implicit, time difference scheme, numerical experiment, computational stability

## I. INTRODUCTION

Spectral models are developed so rapidly that they become the powerful tools widely employed in the following fields of weather, climatic variation and operational meteorological services, one major reason just results from successfully adopting semi-implicit time difference scheme.

Compared with the time implicit scheme, the semi-implicit time difference scheme not only obviously enlarges time-integration step, but also greatly facilitates the computation of all spectral coefficients in prognostic equations without any numerical iteration, especially in Helmholtz divergence equation. In other words, utilizing semi-implicit time difference scheme in a spectral model is very convenient and only needs a little of additional numerical computation costing.

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Based on the principle dividing atmospheric evolutions into both faster and slower processes. Zeng (1961) firstly proposed semi-implicit time difference scheme. and pointed out that the time step of numerical integration can be easily enlarged with a method employing implicit- and explicit-time difference on linear terms (closely related to the augmentation of high-frequency gravity waves) and nonlinear terms (related to those slower atmospheric evolution processes. for example. advection terms). respectively. However. because the above paper is formally published in Russian. the semi-implicit time difference scheme does not attract scientists' attention widely until similar schemes are proposed by Robert (1968a: 1968b). After developed by many meteorologists. the semi-implicit time difference scheme now becomes extensively used difference methods in numerical weather prediction. Studies on successfully applied semi-implicit time difference scheme in spectral models can be traced back to the early 1970s (Rebert 1970; Rebert et al. 1972). In those studies the implicit time difference scheme is successfully designed to enlarge the time-integrated step for multi-level baroclinic spectral model of the atmosphere while holding the precision and stability of numerical computation. Further development is carried out by Bourke (1972: 1974). In his papers. the semi-implicit time difference scheme is designed by linearizing those terms related to gravity waves with respect to reference temperatures under the framework of popular scheme of vertical finite difference and variables location. However. from the above scheme proposed by Bourke. it obviously can be seen that the implicit time difference method is not used for all linear terms related to high-frequency gravity waves. and the major purpose should be keeping away from numerical iteration with time integration of spectral coefficients of prognostic equations. Following above studies. many researches have been dealt with the relationship between reference temperature profile and computational stability of semi-implicit time difference scheme of spectral model. For instance. Chen and Ye (1993) investigated the semi-implicit time integration scheme for a reference atmosphere spectral model. especially on computational instability emerged with the excessive bias between reference and real temperature profiles. Because absolute vorticity and divergence are generally used as basic prognostic variables of perfect numerical prognostic equations. the popular semi-implicit time difference schemes of spectral model used today have been changed in form much more than those original schemes. but basic principle of the popular scheme. indeed. is consistent with what proposed by Bourke (1974). In fact. the popular scheme still includes some important linear terms. which are differenced by time explicit scheme and directly related to atmospheric divergence (i. e.. high-frequency gravity waves of atmosphere). This is because that in the interaction terms between absolute vorticity and motion only the interaction between relative vorticity and motion is truly nonlinear. the other part. i. e. the interaction between geostrophic vorticity and motion. is linear and conceals fast wave components related to high-frequency gravity waves of atmosphere. Additionally. due to differenced pressure-gradient force and Coriolis force terms at different time. the popular semi-implicit time difference scheme artificially introduced unbalance while numerically computing their small difference between large quantities. Namely. the semi-implicit time difference just partly applied for general time difference scheme of spectral model. Zheng (1989) has designed one full semi-implicit time difference scheme for a 7-level global spectral model. in which implicit- and explicit-time differences are employed to linear and nonlinear terms, separately. Used this scheme, one can effectively numerically compute the subtle differences between any other two major terms, enhance computational stability of general spectral models, and achieve good results of monthly numerical weather prediction. However, just because of adopting the above full semi-implicit time difference scheme, the time integration of spectral model needs simultaneous spectral prognostic equations and numerical iteration. In some degrees, this restraint counteracts the advantage of enlarging time step of integration using this scheme. Heretofore, further theoretical/practical studies on more suitable semi-implicit time difference scheme are necessary and still remain for general spectral models. Specially we need to go deep into study how to enlarge time integration step (or enhance computational stability under the same time integration step) by properly adopting time difference scheme to those linear terms, which still be included in spectral model and related to activity of atmospheric high-frequency gravity waves, and also need to study how to hold the advantage that all spectral coefficients of prognostic equations, especially of Helmholtz divergence equation, can be worked out without any numerical iteration at the same time.

The purpose of the present paper is to design a more suitable semi-implicit time difference scheme in basic dynamic framework of spectral models according to computational stability analysis conducted to the linear term time difference scheme in simple harmonic motion equation, and apply it into a self-designed spectral dynamic framework by adopting a kind of revised time-explicit-difference scheme to these linear terms still included in general spectral model governing equations. Numerical studies on enlarging (enhancing) time step (computational stability) also have been done under the precondition that all spectral coefficients of prognostic equations, especially of Helmholtz divergence equation, can be computed without any numerical iteration.

## II. DESIGN OF A SEMI-IMPLICIT TIME DIFFERENCE SCHEME OF SPECTRAL MODEL

The numerical algorithm of spectral model usually treats time-tendency of each prognostic variable as summation of several relatively independent parts, and then introduces time-split method to separately integrate atmospheric dynamic processes and physical parameterizations at every time-integrating step. Additionally, physical parameterizations always incorporate into a spectral model in physical grid space. Hence, for both horizontal diffusion and diabatic physical parameterizations, their time-tendency contributions to each prognostic variable can be incorporated into a spectral model separately by utilizing the time-split method. For a spectral atmospheric model using in numerical weather prediction, its adiabatic dynamic framework adopting the time-split method is the core part of spectral method application. Because one dynamic framework of spectral model can perfectly include spectral method and semi-implicit time difference scheme, it is convenient and helpful to give prominence to the research of semi-implicit time difference scheme of spectral model by using well-designed dynamic framework. Relative to any spectral models including more complex thermodynamic processes, using the spectral dynamical framework composed of simplified prognostic equations is no influences on sufficient discussion of semi-implicit time difference scheme. Therefore, utilizing a basic dynamic framework of spectral model, in which all of horizontal diffusion terms have been omitted, will carry out the following discussion. Here, as a matter of convenience, the following specifications are introduced: a double-under-bar " = " represent a matrix, arrows " $\downarrow$ " and " $\rightarrow$ " denote a column and a row vectors over the vertical grids, respectively.

## 1. *The Problem of Popular Semi-Implicit Time Difference Scheme of Spectral Models*

With spherical sigma vertical coordinates, the equations of basic dynamic framework of spectral model can be expressed in the following vectoral form:

vorticity equation:

$$\frac{\partial \eta \downarrow}{\partial t} - Z \downarrow = 0, \tag{1}$$

divergence equation:

$$\frac{\partial D \downarrow}{\partial t} + \nabla^2 E - d \downarrow = - \nabla^2 (\Phi_S + R\underline{\underline{B}}\, T \downarrow + RT_0 \downarrow \ln P_S) - \nabla^2 [R\underline{\underline{B}}\, (T_v - T) \downarrow ], \tag{2}$$

continuity equation:

$$\frac{\partial \ln P_S}{\partial t} - P = \boldsymbol{\pi} \cdot D \downarrow , \tag{3}$$

thermodynamic equation:

$$\frac{\partial T' \downarrow}{\partial t} - T_1 \downarrow - T_2 \downarrow = - \underline{\underline{\tau}} \cdot D \downarrow , \tag{4}$$

humidity equation:

$$\frac{\partial q \downarrow}{\partial t} - Q \downarrow = 0, \tag{5}$$

where $\eta$ is the absolute vorticity and $\zeta = \eta - f$ is the relative vorticity, $D$ is divergence, temperature $T$ (virtual temperature $T_v$) has been divided into two parts $T'$ ($T'_v$) and $T_0$, $T_0$ only is a function of $\sigma$, the others are conventional denotation of spectral model, and the following definitions (6a) — (6i) are introduced:

$$T_1 = - \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda}(UT') - \frac{1}{a} \frac{\partial}{\partial \mu}(VT'), \tag{6a}$$

$$T_2 = DT' - \dot{\sigma} \frac{\partial T'}{\partial \sigma} - \frac{\partial T_0}{\partial \sigma}\left( \sigma \int_0^1 \boldsymbol{V} \cdot \nabla \ln P_S \mathrm{d}\sigma - \int_0^\sigma \boldsymbol{V} \cdot \ln P_S \mathrm{d}\sigma \right) \mathrm{d}\sigma - \frac{RT_0}{c_P \sigma} \int_0^\sigma \boldsymbol{V} \cdot \nabla \ln P_S - \frac{RT'_v}{c_P \sigma} \int_0^\sigma (D + \boldsymbol{V} \cdot \nabla \ln P_S) \mathrm{d}\sigma + \frac{RT_v}{c_P} \boldsymbol{V} \cdot \nabla \ln P_S, \tag{6b}$$

$$Z = \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda} F_v - \frac{1}{a} \frac{\partial}{\partial \mu} F_u, \tag{6c}$$

$$d = \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda} F_u + \frac{1}{a} \frac{\partial}{\partial \mu} F_v, \tag{6d}$$

$$F_u = \eta V - \frac{RT'_v}{a} \frac{\partial}{\partial \lambda} \ln P_S - \dot{\sigma} \frac{\partial U}{\partial \sigma}, \tag{6e}$$

$$F_v = \eta U - \frac{RT'_v}{a} (1-\mu^2) \frac{\partial}{\partial \mu} \ln P_S - \dot{\sigma} \frac{\partial V}{\partial \sigma}, \tag{6f}$$

$$\boldsymbol{\pi} = (\nabla \sigma_1, \nabla \sigma_2, \nabla \sigma_3, \cdots, \nabla \sigma_K), \tag{6g}$$

$$P = - \boldsymbol{\pi} \cdot (\boldsymbol{V} \cdot \nabla \ln P_S) \downarrow , \tag{6h}$$

$$Q = - \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda}(Uq) - \frac{1}{a} \frac{\partial}{\partial \mu}(Vq) + D q - \dot{\sigma} \frac{\partial q}{\partial \sigma}. \tag{6i}$$

Additionally, all horizontal diffusion terms of prognostic equations have been omitted because they can be incorporated into a spectral model separately by utilizing the time-split method as mentioned above.

After adopting the time central difference scheme for time partial differential terms in the left side of Eqs. (1) — (5), the implicit scheme for those terms related to gravity waves in the right side of equations, and the explicit scheme for the other terms, we can derive the popular semi-implicit time difference scheme of spectral models as the following form:

$$\delta_t \eta \downarrow - Z \downarrow = 0, \tag{7a}$$

$$\delta_t D \downarrow + \nabla^2 E - d \downarrow = - \nabla^2 [\Phi_S + \underline{\underline{RB}} (T_v - T) \downarrow ] - \nabla^2 [\underline{\underline{RB}}\, \overline{T'}^t \downarrow + RT_0 \downarrow \overline{\ln P_S}^t ], \tag{7b}$$

$$\delta_t (\ln P_S) - P = - \pi \cdot \overline{D}^t \downarrow , \tag{7c}$$

$$\delta_t T' \downarrow - T_1 \downarrow - T_2 \downarrow = - \underline{\underline{\tau}} \cdot \overline{D}^t \downarrow , \tag{7d}$$

$$\delta_t q \downarrow - Q \downarrow = 0. \tag{7e}$$

Here, the time difference operators $\frac{\partial \Psi}{\partial t} \approx \delta_t \Psi = \frac{(\Psi^{t+\Delta t} - \Psi^{t-\Delta t})}{2\Delta t}$ and $\overline{\Psi}^t = \frac{1}{2}(\Psi^{t+\Delta t} + \Psi^{t-\Delta t})$ are also used, and the other terms at time $t$ omit their superscripts of time.

From the above semi-implicit time difference scheme (7a) — (7e), it can be seen that the absolute vorticity equation (7a) and humidity equation (7e) could be numerically computed separately because they are all independent of the other equations. However, Eqs. (7b) — (7d) constitute a set of three linear simultaneous equations of prognostic variables of divergence, surface pressure and temperature. With elimination method, one can derive the following Helmholtz equation of divergence variable as

$$\begin{aligned} &\left[ \underline{\underline{I}} + \Delta t^2 \frac{n(n+1)}{a^2} ( \underline{\underline{RB}} \cdot \underline{\underline{\tau}} + RT_0 \downarrow \cdot \pi ) \right] (D \downarrow )^{t+\Delta t} \\ &= (D \downarrow )^{t-\Delta t} + 2\Delta t d \downarrow + \frac{n(n+1)}{a^2} \cdot 2\Delta t \times \\ &\left[ \Phi_S \downarrow + E \downarrow + \underline{\underline{RB}} \left( \frac{1}{2} T'^{t-\Delta t} + T'_v - T' \right) \downarrow + \frac{1}{2} \cdot RT_0 \downarrow (\ln P_S)^{t-\Delta t} \right] + \\ &\Delta t \cdot \frac{n(n+1)}{a^2} \underline{\underline{RB}} \left[ T'^{t-\Delta t} + 2\Delta t (T_1 \downarrow + T_2 \downarrow ) - \Delta t \underline{\underline{\tau}} \cdot (D \downarrow )^{t-\Delta t} \right] + \\ &\Delta t \cdot \frac{n(n+1)}{a^2} RT_0 \downarrow \left[ \ln P_S{}^{t-\Delta t} + 2\Delta t \cdot P - \Delta t \pi \cdot (D \downarrow )^{t-\Delta t} \right], \end{aligned} \tag{8}$$

where $\underline{\underline{I}}$ denotes a unit matrix. Obviously, three-variable linear simultaneous equations *

* In practice, the numerical algorithm of semi-implicit time difference of a spectral model is firstly carried out by spectral coefficient prognostic equations, which are derived by using the orthogonality relationship of spherical harmonic function to the prognostic equations expanded in terms of spherical harmonic functions. And then the prognostic variables at next time step are gotten by numerical summation of their spherical harmonic expansion. Because here discussion only focuses on semi-implicit time difference scheme of spectral model, it is feasible and convenient with prognostic equations in the vectorial form previous to spherical harmonic expansion.

(7b) — (7d) can be solved without any numerical iteration in despite of introducing the semi-implicit time difference scheme into divergence equation (7b), temperature equation (7d) and surface pressure equation (7c). After numerically computing $D^{t+\Delta t}$ separately based on Helmholtz equation of divergence (8), $\overline{D}^t$ can be obtained from relationship of $\overline{D}^t = \frac{1}{2}(D^{t+\Delta t} + D^{t-\Delta t})$, and variables $T'^{t+\Delta t}$ and $\ln P_S{}^{t+\Delta t}$ at the next time step can also be obtained by substituting $\overline{D}^t$ into temperature equation (7d) and surface pressure equation (7c), respectively.

As shown in the above semi-implicit time difference scheme, in the left side of equations (7a) — (7e) all terms are differenced by explicit time scheme as nonlinear parts except for time partial differential terms. However, in the interaction terms ($\eta U$ and $\eta V$) between absolute vorticity and motion, only interactions between relative vorticity $\zeta$ and motion, i. e. $\zeta U$ and $\zeta V$, represent the essential nonlinear terms, the other major parts $fU$ and, $fV$ interaction between geostrophic vorticity $f$ and motion, are linear because $\eta = \zeta + f$, $f$ is about 10 times magnitude of $\zeta$ and geostrophic vorticity $f$ is independent of time variable $t$. In fact some major liner terms are still concealed in $Z$ and $d$ in the left side of vorticity equation (7a) and divergence equation (7b).

Introducing the following definitions of nonlinear terms:

$$\overline{F_u} = \zeta V - \frac{RT'_v}{a}\frac{\partial}{\partial\lambda}\ln P_S - \dot{\sigma}\frac{\partial U}{\partial \sigma}, \tag{9a}$$

$$\overline{F_v} = \zeta U - \frac{RT'_v}{a}(1-\mu^2)\frac{\partial}{\partial\mu}\ln P_S - \dot{\sigma}\frac{\partial V}{\partial\sigma}, \tag{9b}$$

and

$$\overline{Z} = \frac{1}{a(1-\mu^2)}\frac{\partial}{\partial\lambda}\overline{F_v} - \frac{1}{a}\frac{\partial}{\partial\mu}\overline{F_u}, \tag{10a}$$

$$\overline{d} = \frac{1}{a(1-\mu^2)}\frac{\partial}{\partial\lambda}\overline{F_u} + \frac{1}{a}\frac{\partial}{\partial\mu}\overline{F_v}, \tag{10b}$$

we can entirely rewrite variables $Z$ and $d$ as linear and nonlinear parts, i. e.,

$$Z = \overline{Z} - fD - \frac{2\Omega}{a}V, \tag{11}$$

$$d = \overline{d} + f\zeta - \frac{2\Omega}{a}U. \tag{12}$$

Of the semi-implicit time difference scheme adopted by spectral models in a worldwide basis, the vorticity equation (7a) and divergence equation (7b) can be expressed as the following alternative forms according to the two relationships above.

$$\delta_t\zeta\downarrow - \overline{Z}\downarrow + \left(fD\downarrow + \frac{2\Omega}{a}V\downarrow\right) = 0, \tag{13a}$$

$$\begin{aligned}\delta_t D\downarrow + \nabla^2 E - \overline{d}\downarrow - \left(f\zeta\downarrow - \frac{2\Omega}{a}U\downarrow\right) = &- \nabla^2[\Phi_S + R\underline{\underline{B}}(T_v - T)\downarrow] - \\ &\nabla^2(R\underline{\underline{B}}\,\overline{T'}\downarrow + RT_0\downarrow\overline{\ln P_S{}'}),\end{aligned} \tag{13b}$$

where the relationship $\frac{\partial\eta}{\partial t} = \frac{\partial}{\partial t}(\zeta + f) = \frac{\partial\zeta}{\partial t}$ has been used.

Above theoretical analyses clearly show that the above semi-implicit time difference scheme has been changed much in form compared with those original ones of spectral models because absolute vorticity and divergence are generally used as basic prognostic variables of perfect numerical prognostic equations, but the basic principle is consistent with that proposed by Bourke (1974). Meanwhile, above analyses also clearly indicate that semi-implicit time difference scheme just partly is incorporated into the popular spectral time difference scheme, in which prognostic equations still include some important linear terms differenced by time explicit scheme and directly related to atmospheric high-frequency gravity waves (involved in atmospheric divergence). Furthermore, the popular semi-implicit time difference scheme artificially introduces unbalance between pressure-gradient force and Coriolis force terms while numerically computing their small difference between large quantities because of two terms differenced at different time. From above analyses, one can also easily find that the spectral coefficient variables of $\zeta$, $D$, $U$ and $V$ expanded in terms of spherical harmonic functions could counterchange by introducing stream and potential functions, and the corresponding spectral coefficients of $f\zeta$, $fD$ could be computed by virtue of the following recurrence relationship of associated Legendre polynomials: ,

$$\mu P_n^m(\mu) = \frac{1}{2n+1}\left[(n+1-m)P_{n+1}^m(\mu) + (n+m)P_{n-1}^m(\mu)\right].$$

But above equations (13a) and (13b) must consist of simultaneous spectral prognostic equations and compute with numerical iteration method if the terms parenthesized in the left side of two equations are directly differenced by implicit time difference scheme. As mentioned in the introduction, Zheng (1989) already successfully designed a full semi-implicit time difference scheme for a global spectral model, in which implicit- and explicit-time differences are employed to linear and nonlinear terms, separately. By virtue of the nature that simultaneous spectral prognostic equations should be computed with numerical iteration, this scheme could effectively overcome the problem about subtle differences between any two major terms. But using this scheme of iteration obviously counteracts the advantage that implicit scheme can enlarge time step of integration to some extent. All factors show that, presupposing the advantage without any numerical iteration for spectral coefficients of prognostic equations, it is worth conducting further theoretical/practical studies on enlarging time integration step (or enhancing computational stability under the same time integration step) of semi-implicit time difference scheme.

2. *Analyses on Computational Stability of Time Difference Scheme*

With particular purpose to the problem of popular semi-implicit time difference scheme of spectral models, we expect that a kind of revised time-explicit-difference scheme could be designed out for these linear terms still included in spectral model. And this revised scheme only need change a little numerical algorithms of the popular scheme so as to efficiently enlarge time integration step (or enhance computational stability). Because the prognostic equations of an atmospheric spectral model are very complex, and it is very difficult to analyze their computational stability directly, in next two sections the following simple harmonic motion equation

$$\frac{\mathrm{d}F}{\mathrm{d}t} = -i\sigma F \tag{14}$$

is used to investigate the application possibility of our revised time-explicit-difference scheme by comparing the computational stability criteria of Eq. (14) that is differenced by following two schemes. One is that the term in the right side of Eq. (14) is directly explicit differenced with the left side term center differenced. The other is the same as the previous one except for differencing the term in the right side of Eq. (14) with our revised explicit time difference scheme.

(1) *Computational stability for directly explicitly differencing the linear term in the right side of Eq.* (14)

In the first scheme the term in the right side of Eq. (14) is directly explicitly differenced with the left side term center difference. the corresponding difference equation of the simple harmonic motion equation (14) is

$$\frac{F(t+\Delta t)-F(t-\Delta t)}{2\Delta t}=i\sigma F(t). \tag{15}$$

Apparently, $F(t)=F(0)e^{-i\sigma t}$ is one analytical solution of Eq. (14). Thus presuming $F(t)=F(0)e^{-i\sigma n\Delta t}$ as one solution of its difference equation (15) and substituting it into Eq. (15), one can derive

$$e^{-i\sigma\Delta t}-e^{i\sigma\Delta t}=-2i\sigma\Delta t. \tag{16}$$

By setting $N=e^{-i\sigma\Delta t}$, Eq. (16) can be rewritten as

$$N^2+2i\sigma\Delta tN-1=0. \tag{17}$$

Its analytical solution should be

$$N=-i\sigma\Delta t\pm\sqrt{1-(\sigma\Delta t)^2}. \tag{18}$$

If $\sigma\Delta t>1$, $1-(\sigma\Delta t)^2<0$ should be true. $N$ should be an imaginary number and its moduli satisfy the following relationship

$$\begin{aligned}|N|&=\left[(\sigma\Delta t)^2\pm2\sigma\Delta t\sqrt{(\sigma\Delta t)^2-1}+(\sigma\Delta t)^2-1\right]^{1/2}\\&\geqslant\left[1\pm2\sigma\Delta t\sqrt{(\sigma\Delta t)^2-1}\right]^{1/2}.\end{aligned} \tag{19}$$

One modulus of them should be larger than real number 1. In this case the numerical computation of Eq. (15) is unstable with the result that $F=F(0)(N)^n$ is increasing with number $n$ enlarging. Contrariwise, if $\sigma\Delta t\leqslant1$, then $N$ should be a complex number and its moduli always have $|N|=1$. Thus the numerical computation of Eq. (15) is stable because its two solutions never increase with number $n$ enlarging. Above analyses show that in the scheme using directly explicit difference for the term in the right side of Eq. (14) and center difference for the left side term, the corresponding computational stability criterion should satisfy the following relationship

$$\sigma\Delta t\leqslant1,\quad \text{i.e.}\quad \Delta t\leqslant\frac{1}{\sigma}. \tag{20}$$

It means that the time integrating step $\Delta t$ must be less than the smallest scale included in time variable $t$.

(2) *Computational stability for differencing the term in the right side of Eq.* (14) *with our revised explicit time difference scheme*

With center time difference scheme for the term in the left side of Eq. (14), and the following difference scheme (21) for the right side linear term,

$$\overline{F}' = F(t) + \Delta t B \cdot \left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)_{[t-\Delta t, t]} = (1 + B)F(t) - BF(t - \Delta t), \tag{21}$$

one can yield the corresponding differenced equation as

$$\frac{F(t + \Delta t) - F(t - \Delta t)}{2\Delta t} = - i\sigma[(1 + B)F(t) - BF(t - \Delta t)], \tag{22}$$

where $B$ is an adjustable parameter. And on the condition of $B=0$, the scheme (21) is identical with the first scheme of directly explicitly differencing the linear term in the right side of Eq. (14).

Analogously, substituting $F(t) = F(0)\mathrm{e}^{-i\sigma n\Delta t}$ into Eq. (22) and conducting simple derivations yield

$$N^2 + 2i\sigma\Delta t(1 + B)N - [1 + 2i\sigma\Delta t B] = 0. \tag{23}$$

Its solutions are

$$N = - i\sigma\Delta t(1 + B) \pm \sqrt{1 - [\sigma\Delta t(1 + B)]^2 + i\sigma 2\Delta t B}. \tag{24}$$

Its computational stability needs $|N| \leqslant 1$.

When $B \neq 0$, Eq. (24) is a complex number in the following form:

$$M = ci + \sqrt{a + bi}. \tag{25}$$

Its moduli satisfy

$$\begin{aligned} |M| &= \{[c + r^{1/2}\sin\beta]^2 + [r^{1/2}\cos\beta]^2\}^{1/2} \\ &= \{c^2 + 2cr^{1/2}\sin\beta + r\}^{1/2} \leqslant \{c^2 + 2cr^{1/2} + r\}^{1/2} = |c + r^{1/2}|, \end{aligned} \tag{26}$$

where $r = \{a^2 + b^2\}^{1/2}$, $\theta = \mathrm{arctg}\,\frac{a}{b}$, $\beta = \frac{\theta + 2k\pi}{2}$, $k = 0, 1, 2, \cdots$. From Eq. (26), one has $|M| \leqslant 1$ if only $c + r^{1/2} \leqslant 1$.

In Eq. (25) setting, $M = N$, $c = -\Delta t\sigma(1+B)$, $a = 1 - [\Delta t\sigma(1+B)]^2$, $b = 2\Delta t\sigma B$, and substituting them into the relationship $c + r^{1/2} \leqslant 1$, one has

$$r^2 \leqslant (1 - c)^4. \tag{27}$$

If the following relationship

$$\{1 - [\Delta t\sigma(1 + B)]^2\}^2 + (\sigma 2\Delta t B)^2 \leqslant [1 + \Delta t\sigma(1 + B)]^4$$

is true, i. e. relationships

$$(\sigma 2\Delta t B)^2 \leqslant [1 + \Delta t\sigma(1 + B)]^2 \cdot \{[1 + \Delta t\sigma(1 + B)]^2 - [1 - \Delta t\sigma(1 + B)]^2\}$$

and

$$\Delta t\sigma \leqslant \frac{(1 + B) \cdot [1 + \Delta t\sigma(1 + B)]^2}{B^2} \tag{28}$$

hold, then we can have $|N| \leqslant 1$ and stable numerical computation. Relationship (28) is the computational stability criterion of Eq. (22). It shows the inter-constraint-relation among integrating time step $\Delta t$, wave frequency $\sigma$ and the adjustable parameter $B$ when Eq. (22) is stable with time integration.

Obviously, If $B \geqslant -1$, we have $1 + B \geqslant 0$ and $1 + \Delta t\sigma(1 + B) \geqslant 0$. From relationship (28) we have

$$\Delta t\sigma \leqslant \frac{(1 + B) \cdot [1 + \Delta t\sigma(1 + B)]^2}{B^2} \leqslant \frac{(1 + B)}{B^2}. \tag{29}$$

Therefore, if only

$$\Delta t \leqslant \frac{(1 + B)}{B^2} \cdot \frac{1}{\sigma} \tag{30}$$

and $B \geqslant -1$, $B \neq 0$ hold at the same time, the relationship $|N| \leqslant 1$ is always true. The difference scheme (22) can satisfy the requirement of computational stability.

Comparing relationship (30) with (20), one can easily find that only if

$$\frac{(1+B)}{B^2} \geqslant 1 \tag{31}$$

holds, using the second difference scheme (22) can lose the restriction of computational stability in relation to using the first scheme (15). And then it can enlarge the time step of numerical integration. The reasonable range of $B$ in Eq. (31) should satisfy

$$-1 \leqslant B \leqslant \frac{1+\sqrt{5}}{2} \text{ and } B \neq 0. \tag{32}$$

For example, setting $B=1$, which is identical with differencing the linear term of the equation with a $2F(t)-F(t-\Delta t)$ time smooth operator. According to relationship (20), if only

$$\Delta t \leqslant \frac{(1+B)}{B^2}\frac{1}{\sigma} = \frac{2}{\sigma} \tag{33}$$

satisfies, the second difference scheme (22) should be of numerical computational stability. Furthermore, compared to the first scheme (15), theoretically it can still hold the numerical computational stability after the maximum time step of scheme (15) is doubled.

Based on the fact that the general numerical algorithms of spectral models still include some important linear terms which are differenced by time explicit scheme and directly related to atmospheric divergence (i. e., high-frequency gravity waves of atmosphere), the comparative analyses of the previous two sections indicate that properly incorporating the revised time-explicit-difference scheme (21) into these linear terms of spectral model should make the computational costing more economical. Meanwhile it still holds the advantage that all spectral coefficients of prognostic equations can be numerically computed without any iteration. The following design of semi-implicit time difference scheme of spectral model is just based on the above two specific characteristics.

(3) *Design of an improved semi-implicit time difference scheme*

According to the previous discussion, we design an improved semi-implicit time difference scheme in our basic dynamic framework of spectral model as the following vectorial form:

$$\delta_t \eta \downarrow - \overline{\overline{Z}} \downarrow = 0, \tag{34a}$$

$$\delta_t D \downarrow + \nabla^2 E - \overline{\overline{d}} \downarrow = -\nabla^2[\Phi_S + R\underline{\underline{B}}(T_v - T)\downarrow] - \nabla^2(R\underline{\underline{B}}\,\overline{T'}\downarrow + RT_0\downarrow \overline{\ln P_S'}), \tag{34b}$$

$$\delta_t(\ln P_S) - P = -\pi \cdot \overline{D'} \downarrow, \tag{34c}$$

$$\delta_t T' \downarrow - T_1 \downarrow - T_2 \downarrow = -\underline{\underline{\tau}} \cdot \overline{D'} \downarrow, \tag{34d}$$

$$\delta_t q \downarrow - Q \downarrow = 0. \tag{34e}$$

Here, all definitions are same as the previous ones, terms at time $t$ also omit their superscripts of time. And with scheme (22), the term $\overline{\overline{Z}}$ in the left side of the vorticity equation (34a) is differenced timely as the following form:

$$\bar{\bar{Z}} = \frac{1}{a(1-\mu^2)}\frac{\partial}{\partial\lambda}\left(\zeta U + f\bar{\bar{U}}' - \frac{RT'_v}{a}(1-\mu^2)\frac{\partial}{\partial\mu}\ln P_s - \dot{\sigma}\frac{\partial V}{\partial\sigma}\right) - \frac{1}{a}\frac{\partial}{\partial\mu}\left(\zeta V + f\bar{\bar{V}}' - \frac{RT'_v}{a}\frac{\partial}{\partial\lambda}\ln P_S - \dot{\sigma}\frac{\partial U}{\partial\sigma}\right) = Z - B\left[f(D - D^{t-\Delta t}) + \frac{2\Omega}{a}(V - V^{t-\Delta t})\right], \tag{35a}$$

where $B$ is a selectable and adjustable parameter, and can be defined through numerical experiments. Similarly, in the left side of the divergence equation (34b) the term $\bar{\bar{d}}$ is differenced timely as the following form:

$$\bar{\bar{d}} = \frac{1}{a(1-\mu^2)}\frac{\partial}{\partial\lambda}\left(\zeta V + f\bar{\bar{V}}' - \frac{RT'_v}{a}\frac{\partial}{\partial\lambda}\ln P_S - \dot{\sigma}\frac{\partial U}{\partial\sigma}\right) + \frac{1}{a}\frac{\partial}{\partial\mu}\left(\zeta U + f\bar{\bar{U}}' - \frac{RT'_v}{a}(1-\mu^2)\frac{\partial}{\partial\mu}\ln P_S - \dot{\sigma}\frac{\partial V}{\partial\sigma}\right) = d + B\left[f(\zeta - \zeta^{t-\Delta t}) - \frac{2\Omega}{a}(U - U^{t-\Delta t})\right]. \tag{35b}$$

With the above two relationships, the semi-implicit difference scheme (34a) – (34e) can be expressed as the following algorithms that are more easily to do numerical computation:

$$\delta_t\eta\downarrow - Z\downarrow + B\left[f(D-D)^{t-\Delta t}\downarrow + \frac{2\Omega}{a}(V - V^{t-\Delta t})\downarrow\right] = 0, \tag{36a}$$

$$\delta_t D\downarrow + \nabla^2 E - d\downarrow - B\left[f(\zeta-\zeta)^{t-\Delta t}\downarrow - \frac{2\Omega}{a}(U - U^{t-\Delta t})\downarrow\right] = -\nabla^2[\Phi_S + R\underline{\underline{B}}(T_v - T)\downarrow] - \nabla^2(R\underline{\underline{B}}\,\overline{T}'\downarrow + RT_0\downarrow\overline{\ln P_S'}), \tag{36b}$$

$$\delta_t(\ln P_S) - P = -\pi\cdot\overline{D}'\downarrow, \tag{36c}$$

$$\delta_t T'\downarrow - T_1\downarrow - T_2\downarrow = \underline{\underline{\tau}}\cdot\overline{D}'\downarrow, \tag{36d}$$

$$\delta_t q\downarrow - Q\downarrow = 0. \tag{36e}$$

Compared the improved scheme (34) with the scheme (7) widely used in numerical algorithms of spectral models, it is significantly different that the linear parts of $-fD-\frac{2\pi}{a}V$ and $-f\zeta-\frac{2\Omega}{a}U$ have been separated from the dynamic advection terms in the general difference equations of vorticity and divergence. And those linear parts are differenced with the scheme (21) after introducing an adjustable parameter $B$. While the relationship $B=0$ holds, the improved scheme (34) can be fully reduced back to the popular scheme (7). It is not only helpful to numerical comparative analyses of two semi-implicit time difference schemes of spectral model, but also helpful to make choice of the adjustable parameter $B$ through numerical experiments. In addition, it should be noted that the Helmholtz equation of divergence variable of the improved scheme is

$$\left[\underline{\underline{I}} + \Delta t^2\frac{n(n+1)}{a^2}(R\underline{\underline{B}}\,\underline{\underline{\tau}} + RT_0\downarrow\cdot\pi\right](D\downarrow)^{t+\Delta t} = (D\downarrow)^{t-\Delta t} + 2\Delta t d\downarrow + 2\Delta t B\left\{[f(\zeta-\zeta)^{t-\Delta t}]\downarrow - \left[\frac{2\Omega}{a}(U-U)^{t-\Delta t}\right]\downarrow\right\} + \frac{n(n+1)}{a^2}\cdot 2\Delta t. \left[\Phi_S\downarrow + E\downarrow + R\underline{\underline{B}}\left(\frac{1}{2}T'^{t-\Delta t} + T'_v - T'\right)\downarrow + \frac{1}{2}\cdot RT_0\downarrow[(\ln P_S)]^{t-\Delta t}\right] + \tag{37}$$

$$\Delta t \cdot \frac{n(n+1)}{a^2} R\underline{\underline{B}} \cdot [(T')^{t-\Delta t} + 2\Delta t(T_1 \downarrow + T_2 \downarrow) - \Delta t \underline{\underline{\tau}} \cdot (D \downarrow)^{t-\Delta t}] +$$

$$\Delta t \cdot \frac{n(n+1)}{a^2} RT_0 \downarrow [(\ln P_S)^{t-\Delta t} + 2\Delta tP - \Delta t\pi \cdot (D \downarrow)^{t-\Delta t}].$$

And the improved semi-implicit scheme still has the advantage that all spectral coefficients can be worked out without any numerical interation.

## III. RESULTS OF NUMERICAL EXPERIMENTS

Utilizing a T21L5 global spectral model dynamic framework that includes the improved semi-implicit time difference scheme by introducing some terms related to an adjustable parameter of time integration. (This framework is designed ourselves on the basis of some important globe integral relations such as energy conservation, and is compatible well with the common numerical algorithms of the spectral dynamic framework used internationally. Moreover, it incorporates the role of model topography and includes all numerical algorithms only except for diabatic physic parameterizations. In another paper it is going to be published in detail.) 32-day-integration numerical experiments have been carried out to predict the July monthly mean conditions of general atmospheric circulation with National Meteorological Center (Beijing, China) objective analysis data at 12 GMT 29 June 1995 as initial fields. According to results of those experiments, studies are conducted with two specific purposes. Firstly, comparing results with different adjustable parameter $B$ we investigate the effects of the improved scheme on computational precision of dynamical framework of a spectral model after effectively overcoming the defect of commonly used time difference scheme of spectral model only partly using semi-implicit integrating scheme; Secondly, under the precondition of keeping the same computational precision as that $B = 0$ case, we also investigate effects of introducing adjustable parameter $B$ of the improved scheme on computational stability and costing.

### 1. *Effects on Precision of Monthly Numerical Weather Prediction*

In the section, with the same numerical integration time step of 45-min numerical experiments are carried out, separately, for the following conditions $B=0$, 0.10, 0.15, 0.20 and 0.25. After integrated the case introduced in previous paragraph, and compared the anomaly correlation coefficients ($RT$, $RU$ and $RV$) of monthly mean fields of temperature, zonal wind component and meridional wind component at 500 hPa level in which U.S. NCEP monthly mean re-analyzed data 49-year-averaged from 1950 to 1998 are used as the corresponding climatic monthly mean observations, the following results could be achieved. Out of four numerical experiments, the highest verified values are achieved in the case $B=0.15$. The 500 hPa $RT$, $RU$ and $RV$ values are 0.26, 0.27 and 0.35 over the whole globe, respectively. Over the Northern Hemisphere their values are also up to 0.49, 0.28 and 0.41, respectively. For the case $B=0$, however, $RT$, $RU$ and $RV$ values are 0.26, 0.27 and 0.32 over the whole globe, and 0.49, 0.30 and 0.40 over the Northern Hemisphere, respectively. The result of the former case is slightly better than the latter case. It suggests that with a suitable adjustable parameter $B$ the improved scheme could overcome the defect of popular time difference scheme of spectral model only partly using semi-imlicit difference scheme to some extent. And under the same time step of numerical integration as the commonly used semi-implicit scheme. adopting the improved scheme can more efficiently prevent fast internal- and external-gravity waves from abrupt increment so as to achieve better precision of numerical weather prediction.

2. *Effects on Time Step of Numerical Integration of Spectral Model*

With the same adjustable parameter $B=0.15$ monthly numerical experiments are carried out for time steps 50, 55. 60. 65 and 70 min., respectively. According to those numerical results we could investigate the effects of introducing adjustable parameter $B$ of the improved scheme on computational stability and costing under the precondition of keeping the same computational precision as that $B=0$ case. Results show that all 32-day experiments are of computational stability except that integration of the last one with time step 70 min. crashes down on the 13th model day. Furthermore. out of the first four stable numerical experiments. the results with time step 55 min., referred to as Exp. T1. is the best of all. Therefore. in this paper its results are presented in Table 1 in detail. To make comparative analyses. another numerical experiment, referred to as Exp. T2, is also done with same time step 55 min. while the adjustable parameter $B$ is 0.

**Table 1.** 500 hPa Monthly Mean Anomaly Correlation Coefficients of Experiments T1 and T2

| Case | Time step of integration | Verified month | Verified area | Numerical experiment | Value of adjustable parameter $B$ | Monthly mean anomaly correlation coefficients | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | $R_T$ temperature | $R_U$ zonal wind | $R_V$ meridional wind |
| 29 June 1995 | 55 minutes | July 1995 | Globe | T1 | 0.15 | 0.26 | 0.31 | 0.34 |
| | | | | T2 | 0.00 | 0.24 | 0.24 | 0.25 |
| | | | Northern Hemisphere | T1 | 0.15 | 0.48 | 0.34 | 0.42 |
| | | | | T2 | 0.00 | 0.48 | 0.29 | 0.44 |

The corresponding climatic monthly mean observations are average values of U. S. NCEP 49-year (1950—1998) monthly mean re-analyzed data.

Table 1 shows that over the whole globe $R_T$. $R_U$ and $R_V$ values of Exp. T1 are 0.26. 0.31 and 0.34. respectively. For the case with $B=0$ and time step 45 min. however. its $R_T$. $R_U$ and $R_V$ values are 0.26. 0.31 and 0.32, respectively. There are only small differences between results of above two numerical experiments over the whole globe. Furthermore. over the Northern Hemisphere differences between $R_T$. $R_U$ and $R_V$ values also have the similar characteristics to those of globe (referred to the previous section). Above comparison shows that applying the improved semi-implicit time difference scheme with $B=0.15$ and 55 min. time step still could work out the pretty much the same numerical results as those of integration applying the commonly used semi-implicit scheme of spectral model with time step 45 min. As the 32-day case stands. Exp. T1 could save lots of computational costing compared to the commonly used scheme with 45 min. time step. With a personal computer, the former time costing is lessened 24 min. than the latter. Moreover, it is also evident that most values of $R_T$, $R_U$ and $R_V$ of Exp. T1 are higher than Exp. T2. It is consistent well with the above analyses that predicting precision could be improved by introducing a suitable adjustable parameter $B$.

## IV. CONCLUSIONS

Based on the computational stability analysis conducted to the linear term time difference scheme in simple harmonic motion equation, an improved semi-implicit time difference scheme is designed in this paper. With an own-designed T21L5 global spectral dynamic framework primary numerical experiments have also been carried out to verify practicability of the improved scheme. After the comparative analyses the following conclusions can be summarized.

(1) Theoretical analyses indicate that the semi-implicit time difference is just partly applied in the general time difference scheme of spectral models. And the popular semi-implicit time difference scheme still includes some important linear terms differenced by time explicit scheme. Furthermore, these major terms are directly related to fast internal- and external-gravity waves of atmosphere (related to divergence terms). Additionally, due to using time difference on two terms at different time, the popular scheme artificially introduces unbalance between pressure-gradient force and Coriolis force terms while numerically computing their small difference between large quantities.

(2) In our study, an improved semi-implicit scheme of spectral model is designed by introducing a suitable adjustable parameter and adopting a kind of revised time-explicit-difference scheme to these linear terms still included in spectral model governing equations, and it can enlarge (enhance) the time-step (computation stability) under the precondition that all spectral coefficients of prognostic equations, especially of Helmholtz divergence equation, still can be computed without any numerical iteration.

(3) Numerical results of one case prove that under the same time step of numerical integration as the commonly used semi-implicit scheme, adopting the improved scheme not only can more efficiently prevent fast internal- and external-gravity waves from abrupt increment, but also can make the smaller difference between major terms of pressure-gradient force and Coriolis force more precise. Additionally, with an introduced adjustable parameter 0.15 the time step of the improved scheme could be enlarged by 10 min. more than the commonly used scheme while they work out the pretty much the same monthly numerical results.

(4) For a spectral atmospheric model using in numerical weather prediction, its adiabatic dynamic framework adopted the time-split method is the core part of spectral method application, and can perfectly include spectral method and semi-implicit time difference scheme. Using a spectral dynamical framework composed of simplified prognostic equations has no significant influences on discussion of semi-implicit time difference scheme of spectral model. Results achieved in this paper are still suitable and general to any spectral model including more complex thermodynamic processes.

(5) The further studies on the improved semi-implicit time difference scheme of spectral model should be done by incorporating it into a diabatic spectral model with perfect physic parameterization.

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